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ON

ELEMENTARY STATICS

FOR THE USE OF SCHOOLS & COLLEGES

BY

W. J. DOBBS, M.A.

SOMETIME FOUNDATION SCHOLAR OF ST. JOHN'S COLLEGE, CAMBRIDGE

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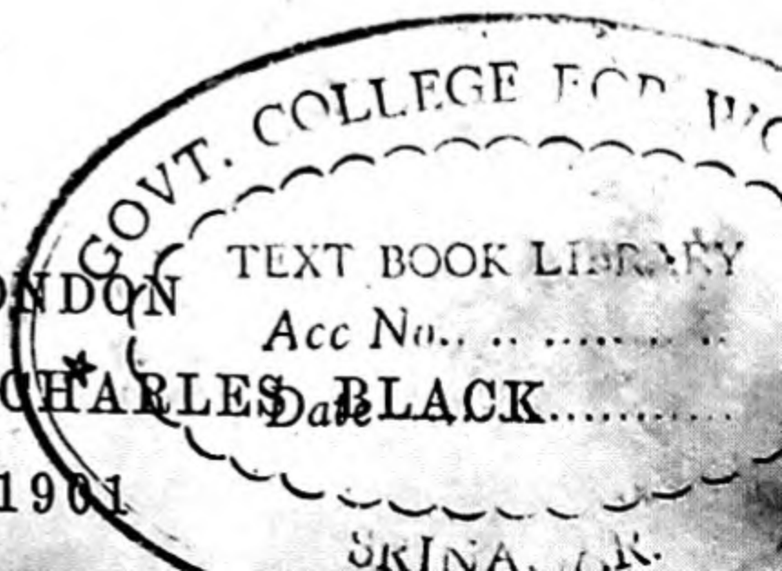

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PREFACE

THE present volume is an Elementary Treatise on the Statics of Coplanar Forces, and is intended for the use of Higher Forms in Schools and Junior Classes at the Universities.

The special feature of the work is its continual insistence upon the double aspect of the subject, namely the geometrical and the analytical. I have in the whole development of the subject allowed Geometry to play its part side by side with Analysis, and I venture to hope that the book will thus cultivate greater resourcefulness than would come from the usual almost exclusive use of the analytical method.

The examples are very numerous, and are mostly of an easy type. The more difficult will be found in the sets which come at the ends of the different chapters. They include, among others, practically all the examples which have been set during the last ten years in the Military Entrance Examinations, together with a large number of original examples accumulated during the same period. The Miscellaneous Examples at the end of the book are intended primarily for the use of Scholarship Candidates; they consist largely of

recent scholarship questions, and require some mathematical power for their solution, but are not beyond the scope of the book.

It is suggested that the portions marked with an asterisk may be omitted by beginners on a first reading.

My thanks are due to several Cambridge friends, and particularly to Mr. R. W. Bayliss, M.A., who has kindly read through the whole of the proof-sheets, and to whom I am indebted for many valuable criticisms and suggestions.

Corrections or suggestions for improvement from teachers or students will be much esteemed.

W. J. DOBBS.

5 Lexham Gardens, W.,
August 15, 1901.

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ELEMENTARY STATICS

CHAPTER I

INTRODUCTION

1. **Matter** *is the substance of which all created things are composed.*

Matter, in some form or other, is continually affecting our senses. It is matter which moves, and its presence is attested by its powers of offering resistance.

2. *A* **Material Body** or simply *a* **Body** *is a limited quantity of matter occupying a limited portion of space.*

In a body matter assumes some definite form in which the dimensions are limited in all directions.

3. *The* **Volume** *of a body is the amount of space it occupies.*

Two bodies may occupy equal portions of space and yet contain unequal quantities of matter, the matter being more closely packed in the one case than in the other.

4. *The* **Mass** *of a body is the quantity of matter it contains.*

5. *A* **Particle** *is a material point.*

In a particle the dimensions are indefinitely small in all directions.

A body may be regarded as an aggregation of an indefinitely large number of indefinitely small portions of matter called *particles*. In this sense a particle is a body of indefinitely small volume and of indefinitely small mass.

The term particle is, however, used in another sense. It frequently happens that the dimensions of a body may be neglected in comparison with those of neighbouring objects, while its mass is of appreciable magnitude. Under these circumstances, and for the purposes of our investigations, the body may be regarded as an appreciable quantity of matter condensed

into an indefinitely small compass, possessing mass and position only. Of course such a portion of matter does not exist in nature, but is an ideal conception. In considering the motion of such a body we lose sight of any possibility of its rotating, and imagine it to describe a single line.

6. *A* **Rigid Body** *is one in which the different particles of which it is composed are so inseparably connected, that they retain the same positions with regard to one another under all circumstances.*

This is again an ideal conception, not occurring in nature.

7. **Force** *is anything which produces or tends to produce motion in matter at rest, or which modifies or tends to modify motion already existing in matter.*

A force, in its strictest sense, is that which produces or tends to produce motion *in a particle* at rest, or that which modifies or tends to modify motion already existing *in a particle.*

A particle, being of indefinitely small mass, the forces which act upon it are also of indefinitely small magnitudes. The forces, however, which we shall consider are in general ideal forces of appreciable magnitude equivalent to an indefinitely large number of the small forces which actually exist.

In the case of a string in a state of tension, the action of one part of the string upon another consists of an aggregation of small forces distributed all over the section of the string. We shall, however, regard them as a single force, applied at a point of the string along a single line, situated in the middle of the string. Thus, if force is applied to a body by means of a string which is pulled, we shall regard the part of the body to which the string is attached as a *geometrical point*, and we shall consider the force as of finite magnitude applied at that point in a geometrical line.

8. *The* **Line of Action** *of a force is the line drawn through its point of application in the direction in which the force acts.*

A force is said to *act along* its line of action.

9. **Statics** *is the science which deals with the conditions under which the forces applied to a body do not disturb its position of rest.*

Under these circumstances the forces are said to be *in equilibrium.*

10. *The* **Weight** *of a body is the force with which the earth attracts it.*

The earth attracts every particle of which a body is com-

posed with a force dependent upon the mass of the particle and its distance from the earth's centre. For particles at the same distance from the centre of the earth, these forces are proportional to the masses of the particles.

The bodies with which we shall have to deal are all small in comparison with the size of the earth, while the distances of their different parts from the centre of the earth are all large, and may be regarded as equal. Thus the attraction of the earth upon a body may be regarded as consisting of a large number of small parallel forces distributed all over the body, each of these forces being applied to a single particle of the body and in magnitude proportional to the mass of the particle. When the body is moved farther away from the earth, these forces are all diminished, but in the same ratio.

We shall regard all these forces as together equivalent to a single ideal force equal to the total weight of the body and applied at some one point of the body.

It will be proved later on that in the case of a rigid body the total weight may be regarded as applied at one point, fixed with regard to the different parts of the body however the body be turned about.

In the case of two different bodies, situated so that their distances from the centre of the earth may be regarded as equal, their total weights are proportional to their masses. Thus we may say that the weight of m units of mass is m times the weight of one unit of mass.

In the present work the unit of mass generally chosen is one pound, and forces are generally measured in terms of the weight of one pound. It must not, however, be forgotten that the weight of one pound, though suitable in most cases for a statical unit of force, is not of constant magnitude, but depends upon the position of the pound relatively to the earth.

11. Action and Reaction.—Forces always occur in pairs, and the association of two masses is necessary for the existence of force. By the mutual association of two masses A and B, *action* is set up between them, and the action of A upon B is equal and opposite to that of B upon A. The action of B upon A may also be regarded as the resistance which B offers to the action of A upon B, and in this aspect it is called the *reaction* of B. We take it as a fundamental axiom that

Action and Reaction are equal and opposite.

Fig. 1 represents two bodies A and B resting upon the earth at H and K, and against one another at L.

The pressure of B upon A at L is equal and opposite to that of A upon B at L.

The pressure of A upon the earth at H is equal and opposite to that of the earth upon A at H.

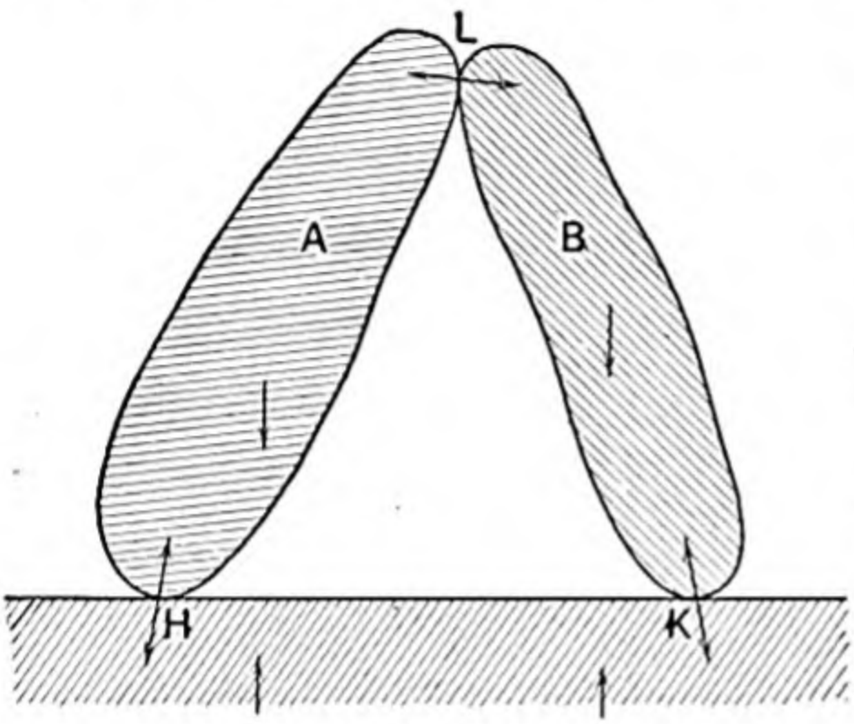

FIG. 1.

The pressure of B upon the earth at K is equal and opposite to that of the earth upon B at K.

The attraction of the earth upon A is equal and opposite to that of A upon the earth.

The attraction of the earth upon B is equal and opposite to that of B upon the earth.

In the present work, except when it is expressly stated to the contrary, we shall consider that all forces acting on a body or particle are applied in one plane.

12. Equilibrium of Two Forces.—We take it as axiomatic that two *equal* forces acting in opposite directions at two points A and B of a rigid body, so that the force acting at A is in direction AB, and that acting at B in direction BA, produce no effect upon the body as a whole. The tendency is merely to throw the portions of the body between A and B into a state of compression, and, as we are dealing with an ideally rigid body, the effect upon the body as one solid piece is *nil*. Similarly, if the two equal and opposite forces act outwards, instead of inwards, they produce no effect upon the body as a whole.

The converse of this principle is equally important; namely, two forces acting upon a rigid body cannot balance one another unless they are equal and opposite and act along the same straight line. Unless they satisfy all three of these conditions the two forces will produce a motion of the body.

We assume, then, the following fundamental axioms:—

If two forces, acting on a rigid body, are equal and act in opposite directions along the same straight line, they are in equilibrium.

Conversely *If two forces, acting upon a rigid body, are in equilibrium, they must be equal and act in opposite directions along the same straight line.*

13. Principle of Transmissibility of Force.—*If a force acts at a point A of a rigid body in a direction from A towards B, and if any other point A′ is taken in AB, or AB produced either way, then the force may be considered to act at A′, instead of at A, and in the same direction as before.*

This follows from the first of the axioms stated above.

For, take any other point A'' in AB, or AB produced either way. At A' and A'' apply two forces F' and F'' respectively, the force F' being in the same direction as the force F, and the force F'' in the opposite direction, and both of the same intensity as the original force F. These two forces F' and F'', being equal and opposite and along the same straight line, themselves form a system in equilibrium, and can be introduced without having any effect upon the body as a whole.

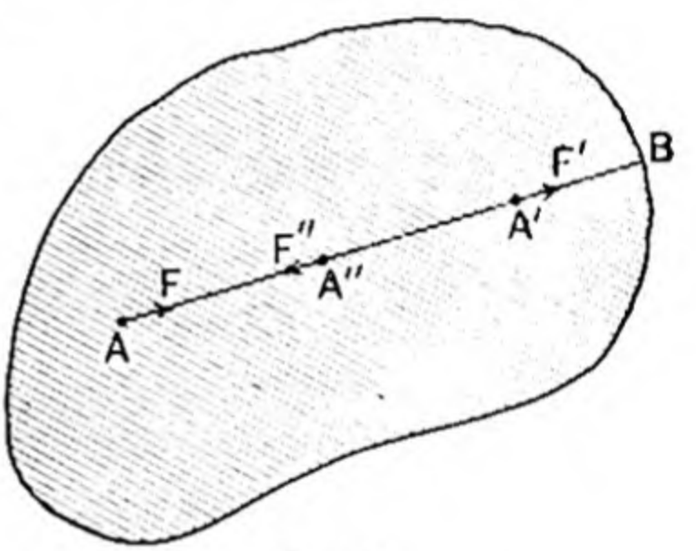

FIG. 2.

The two forces F at A and F'' at A'' may now be removed, for they also are equal and opposite, and act along the same straight line.

We are now left with the force F' at A'.

Thus the force F' at A' may be considered as equivalent to the force F at A, so far as its effect upon the body as a whole is concerned.

Thus there are a succession of points of the body, all situated in the line AB, any one of which may be considered to be the point of application of the force F. This justifies the expression *line of action* as applied to the line AB.

Further, the line of action may be extended beyond the limits of the body, and the force considered as applied at a point outside the actual body altogether, if we suppose the body

to be ideally extended so as to include this point, which must be treated as a point of the body.

14. Converse of the Principle of Transmissibility of Force.—*If a force F acting at A is equivalent to another force F' acting at A' in its effect upon a rigid body as a whole, then the two forces F and F' must be equal, and act in the same direction along the same straight line.*

We can prove this from the second of the two axioms of Art. 12.

Let the force F act at A in the direction from A towards B. In AB take any point A''.

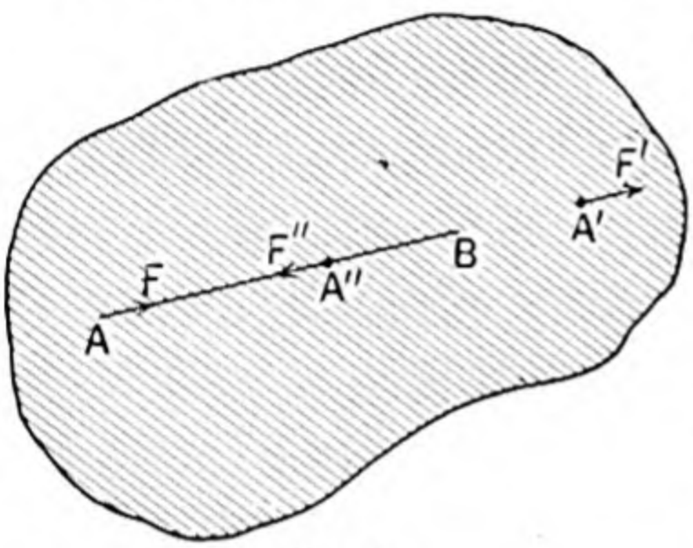

FIG. 3.

Then the force F could be balanced by a force F'' of equal intensity acting at A'' in the opposite direction. But as F' is equivalent to F, it follows that the force F'' would also balance the force F'.

$\therefore$ F'' and F' must be of equal intensity and act in opposite directions along the same straight line.

But F'' and F are of equal intensity and act in opposite directions along the same straight line.

$\therefore$ F' and F must be of equal intensity and act in the same direction along the same straight line, so that A' is a point in AB, or AB produced either way.

15. Resultant.—When a number of forces acting upon a rigid body are in equilibrium, any one may be regarded as balancing the combined effect of the others. But this one force could be balanced by an equal force acting in the opposite direction along the same straight line. Hence we see that, if one of the forces is reversed in direction but remains otherwise unaltered, it may be regarded as equivalent to the remaining forces of the system.

When two or more forces, acting on a rigid body, are in their joint action together equivalent to a single force, that single force is called the **resultant** *of the other forces.*

When two or more forces are regarded as being replaced by their resultant, they are said to be *compounded* into their resultant, and are described as *components* of their resultant.

When one force is regarded as being replaced by two or more forces to which it is equivalent, it is said to be *resolved* into components.

16. Resultant of Forces acting in the Same Straight Line.—If a number of forces act upon a *particle*, all in the same direction, their resultant is clearly equal to their sum.

Further, if the forces act upon the particle, all in the same straight line, but some in one direction and some in the opposite direction, then those in the one direction may be regarded as positive and those in the opposite direction as negative, and the resultant is still the sum of the forces, each having its own proper sign. This we take as self-evident.

Now consider the case of a number of forces acting upon a rigid body all along the same straight line, but applied at different points of that line. By the principle of transmissibility of force, the forces may all be supposed to be transferred to one and the same point in their common line of action. Hence the resultant is again the algebraic sum of the forces.

17. Action of Force upon Bodies not Rigid.—No body in nature is absolutely rigid. When forces are applied to a body at rest, they may produce motion in the body, or they may throw it into a state of strain until the different parts of the body take up a position of equilibrium. In the second case, all parts of the body being now at rest, we may treat the whole body or any portion of it as a rigid body under the influence of the forces which act externally upon that portion.

In order that a body not rigid may be in equilibrium under the action of forces applied to it, so that every portion of the body may be at rest, it is necessary and sufficient that :—

(i) *The forces which act externally upon every conceivable portion of the body should satisfy the conditions of equilibrium of forces acting externally upon an ideally rigid body of the same size and shape.*

(ii) *The external forces which act upon the whole body should not be such as to produce internal actions sufficient to break it.*

The first of these principles shows the practical importance of examining the conditions necessary and sufficient for the equilibrium of an ideally rigid body when acted upon by forces, and enables us to determine, when necessary, the internal actions at different points within a body.

The second principle is also of great practical importance, and belongs to that branch of mechanics which deals with *strength*

of materials. In the present work we shall, in general, assume that every body under consideration is strong enough to bear any strain put upon it.

18. Fine Light Rods and Strings.—By a *light* rod or string we mean one whose weight is inappreciable. By a *fine* rod or string we mean one whose thickness is inappreciable.

Let AB represent a fine light rigid straight *rod* at rest under the influence of two forces, one applied at A and the other at B.

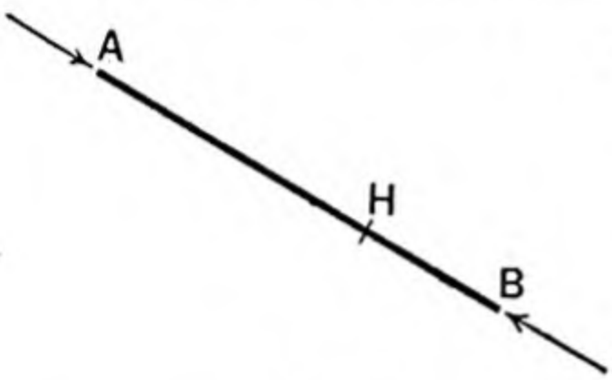

FIG. 4.

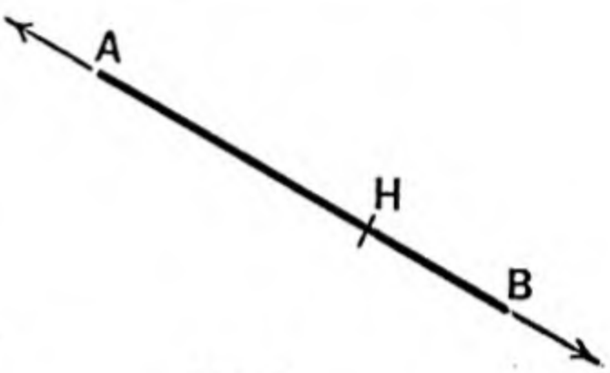

FIG. 5.

We may remove the limitation that the rod should be rigid, and suppose that it has taken up a position of equilibrium under the influence of the forces that act upon it.

These two forces must be equal and opposite, and act along the same straight line. Hence AB must be the line of action of both forces.

The actions upon the rod at A and B must be either both inwards, as in Fig. 4, or both outwards, as in Fig. 5.

For the purpose of determining the stress produced at a point H of the rod situated between A and B, we consider the equilibrium of the portion AH. The external forces which act upon this portion are,—the force applied at A and the action of the adjoining portion HB at H upon the portion AH. These two forces must be equal and opposite and act along the same straight line. Thus, in Fig. 4, the portion HB *presses* against the portion AH in the direction HA with a force equal to the force which acts upon the rod at A. In Fig. 5, the portion HB *pulls* at the portion AH in the direction AH with a force equal to the action at A.

In Fig. 4 the rod is in a state of compression ; such a rod is called a *strut.* In Fig. 5 the rod is in a state of tension ; such a rod is called a *tie.*

In *both* cases the internal action is of the same intensity at every point of the rod, and is in the direction of the rod.

19. If a fine light *string* AB is in equilibrium in a state of tension, under the influence of two forces, one applied at each end, then the two external forces, as in the case of a rod, must be equal and opposite, and act in the same straight line.

Take any point H of the string between A and B, and consider the equilibrium of the portion AH. The external forces which act upon this portion are,—the force applied at A and the pull of the adjoining portion HB at H upon the portion AH. These two forces must be equal and opposite and act along the same straight line. Thus, every point H of the string between A and B is in the line of action of the force applied at A, so that the string is *straight*, and the tension at every point is of the same intensity and equal to the force applied at each end of the string.

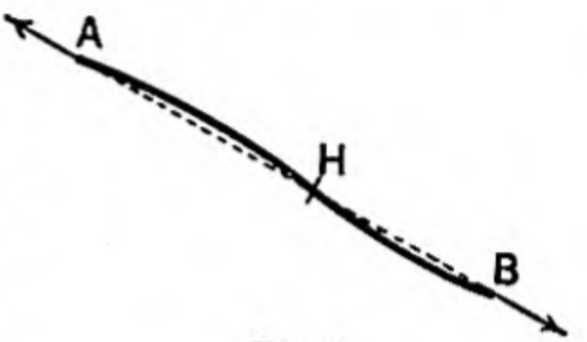

FIG. 6.

The string may be extensible or inextensible. In the first case it will be stretched beyond its natural length before assuming its position of equilibrium.

20. Suppose that a fine light string AB has one extremity A attached to a fixed point, and that it is thrown into a state of tension by means of a force applied at the other extremity B. Then it takes up a position of equilibrium such that the whole string AB is straight and in the direction of the force applied, and the tension of the string at every point is equal to the force applied.

In particular, if a heavy mass is attached at B, then AB takes up the vertical position with B below A, and the tension of the string at every point is equal to the weight of the mass supported.

FIG. 7.

21. We may remove the limitation that the string should be *light*.

Let AB be a fine *heavy* string attached to a fixed point A and supporting a heavy mass at B. Let H be any point of the string between A and B. Then, considering the equilibrium of the system consisting of the portion of string HB and the mass at B, we see that the pull of the string AH at H upon this system is such as to balance its whole weight. Hence the pull of the string AH

upon the string HB is vertically upwards and equal to the sum of the weights of the string HB and the mass at B.

Thus the string rests with B vertically below A, and the tension at any point H is equal to the weight of all below H.

Example.—Suppose that AB is a uniform heavy string of length 12 feet and mass 3 pounds, the mass at B being 40 pounds, and that it is required to determine the tension at a point H, situated 5 feet along the string from A.

As the string is uniform, the mass of any portion is proportional to its length. Hence HB weighs $\frac{7}{12}$ of 3 pounds $= 1\frac{3}{4}$ pounds. Thus the tension at H is the weight of $41\frac{3}{4}$ pounds.

EXAMPLES I

1. A sheet of paper of no appreciable weight, to which is attached a heavy particle at a point P, is supported at a point A. Find the position of equilibrium.

2. A fine light string $ABCD$ is attached to a fixed point A, and supports at B, C, D particles weighing 5, 8, 7 pounds respectively. Find the tensions of the different portions AB, BC, CD of the string.

3. A mass of 10 pounds hangs at the end of a uniform rope 12 feet long, whose other end is fixed. If the rope weighs 3 pounds, find its tension at the point 4 feet from the mass.

4. A uniform heavy string is attached at one extremity to a fixed point and supports at its other extremity a heavy body. Show that the tensions at different points, situated at equal intervals along the string, form an arithmetical progression.

5. A uniform string AB is attached to a fixed point A and supports at its other end B a mass of 60 pounds. The strain on the support at A is 63 pounds' weight, and the tension at a point of the string 5 feet from A is 62 pounds' weight. Find the length of the string and the tension at a point 5 feet from B.

6. A uniform chain AB, of length 7 feet 6 inches and weighing 10 pounds, is attached to a fixed point A, and supports at its other end B a mass of one cwt. If a force of 50 pounds' weight is applied vertically upwards to the chain at a point 5 feet from A, find the tension (i) at a point 4 feet 6 inches from A, and (ii) at a point 6 feet from A.

7. A uniform chain $ABCD$, of length 8 feet 9 inches, and weighing 7 pounds, is attached to a fixed point A, and supports at B, C, D masses of 10, 16, 20 pounds respectively. If AB and BC are of lengths 4 feet and 2 feet respectively, find the tensions at the middle points of AB, BC, CD respectively.

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CHAPTER II

PARALLELOGRAM OF FORCES

22. Geometrical Representation of Force.—We have seen that it is not necessary in general to indicate the actual point of

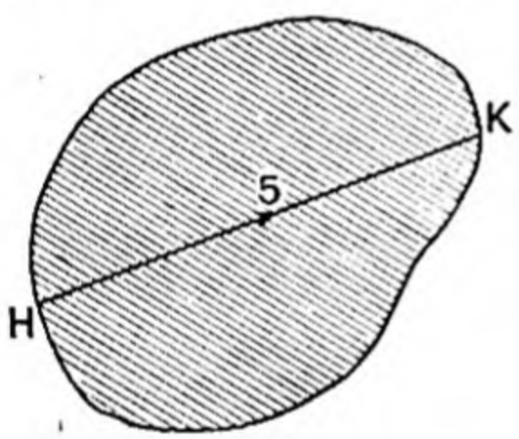

FIG. 8.

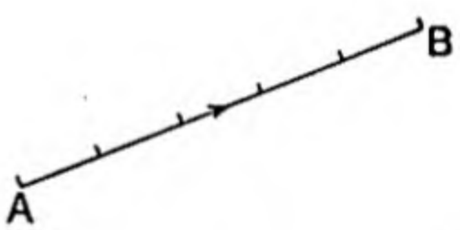

FIG. 8a.

application of a force. The line of action being known, an arrowhead is placed in it, to show the direction in which the force acts along it, and a number placed by the arrowhead indicates the magnitude of the force.

Thus, if the weight of one pound is taken as the unit of force, the number 5 in Fig. 8 indicates that a force of five pounds' weight acts in the direction indicated by the arrowhead along the line *HK*.

The order in which the letters are read is important. If the direction of the force were from *K* towards *H*, we should say that it acted along *KH*.

If the force contained *P* units of force, its magnitude would be denoted by the algebraic symbol *P*. In general, by *a force P* we mean *a force whose measure is P*, i.e. *a force equal to P times the unit of force.*

A straight line can be drawn to show by its length the

magnitude of a force, and by its direction the direction of the force.

Thus, in Fig. 8*a*, the straight line AB is drawn parallel to HK and in the same direction as HK, and, with some suitable scale, AB is taken 5 units of length. In this way the force of 5 pounds' weight, which acts along HK, is represented in magnitude and direction by the line AB.

A force P would be represented by a straight line of length P units drawn in the proper direction.

23. Resultant of Two Forces.—When two forces act upon the same particle in different directions, they tend to move the particle in some definite direction, and it is clear that a single force could be found of proper intensity, which, acting in the opposite direction, would keep the particle at rest. If this force be reversed, it is the equivalent of the other two forces, and may be regarded as capable of replacing the other two.

Thus we see that two forces, acting upon the same particle, *have a resultant.*

Now consider two forces acting upon the same rigid body along intersecting lines. They may be supposed to be transferred so as both to act at the point of intersection of their lines of action. Hence, as before, two forces, acting upon the same rigid body along intersecting lines, *have a resultant*, whose line of action passes through the point of intersection of the lines of action of its component forces.

The following proposition gives the method for determining the resultant of two such forces. A formal proof of the proposition will be found at the end of the present chapter.

24. The Parallelogram of Forces.—*If two forces, acting along OB and OD, are represented in magnitude and direction by OB and OD respectively, and if the parallelogram $OBCD$ is completed, then the resultant acts along the diagonal OC and is represented in magnitude and direction by OC.*

Thus, to find the resultant of two forces P and Q, we first determine the point O where their lines of action intersect. Then from O, along the lines of action of the forces P and Q, we measure OB and OD to contain P units of length and Q units of length respectively. Completing the parallelogram $OBCD$, we draw the diagonal OC and measure it. Suppose that OC contains R units of length. Then the resultant of the two forces P and Q is a force R acting along OC.

The student should notice that the two forces are represented by *OB* and *OD*, both drawn *away from O*, and that the resultant is intermediate in direction to the directions of *P* and *Q*.

The resultant force *R* is not an actual force applied to the

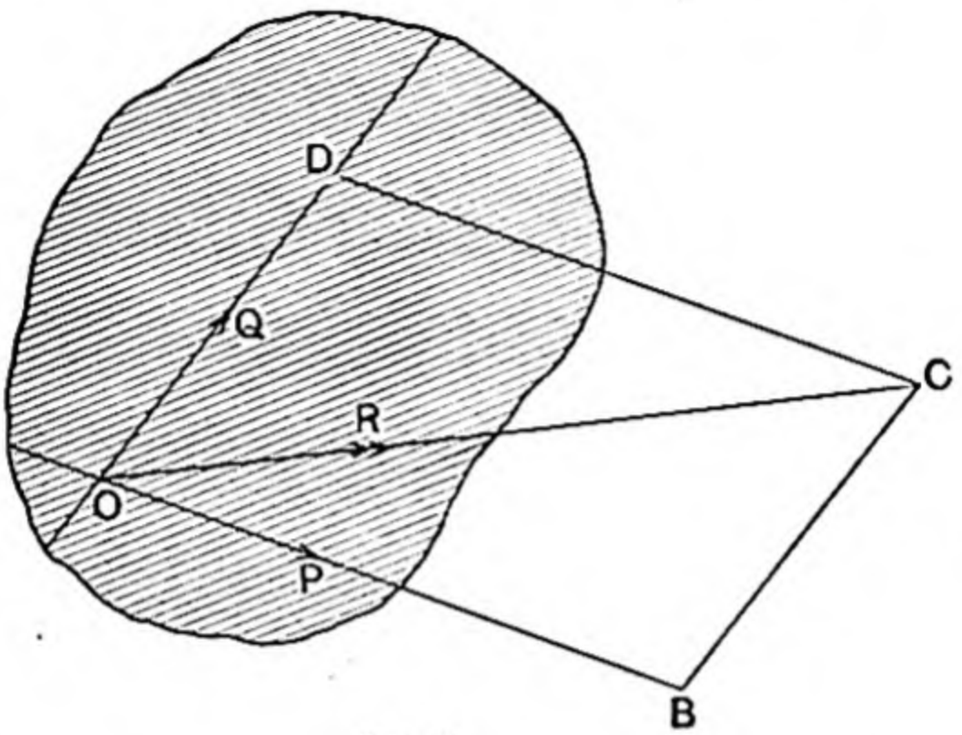

FIG. 9.

body, but an ideal force which may be conceived to replace the two forces *P* and *Q*. Its point of application may be taken to be any point of the body situated in *OC* or *OC* produced either way. If the line of action of the resultant falls wholly outside the body on which the forces act, then we may suppose the body to be ideally extended so as to include some portion of the line *OC*.

25. Experimental Verification of the Parallelogram of Forces.—In Fig. 10, *ABC* represents an endless fine light string, to which three other fine light strings are knotted at *A*, *B*, *C*. The tensions of the three last-mentioned strings are recorded by means of three spring balances shown in the figure.

The points *Q* and *R* are fixed on a board on which a diagram can be drawn, and the whole system is thrown into a state of tension by pulling the string at *A*. The point *P* is then fixed, and when the system has taken up a position of equilibrium, the readings of the balances are found to be (say) *l*, *m*, *n*, as indicated in the figure.

We proceed to make the following construction on the board behind the strings:—Producing *QB* and *RC* to meet at *O*, we measure, on any suitable scale, lines *OM* and *ON* of lengths, *m*

units and n units towards Q and R respectively. Then, completing the parallelogram $OMLN$ and joining OL, it is found that

(i) OL is in the same straight line as the string AP,
(ii) OL and AP are in opposite directions,
(iii) OL contains l units of length.

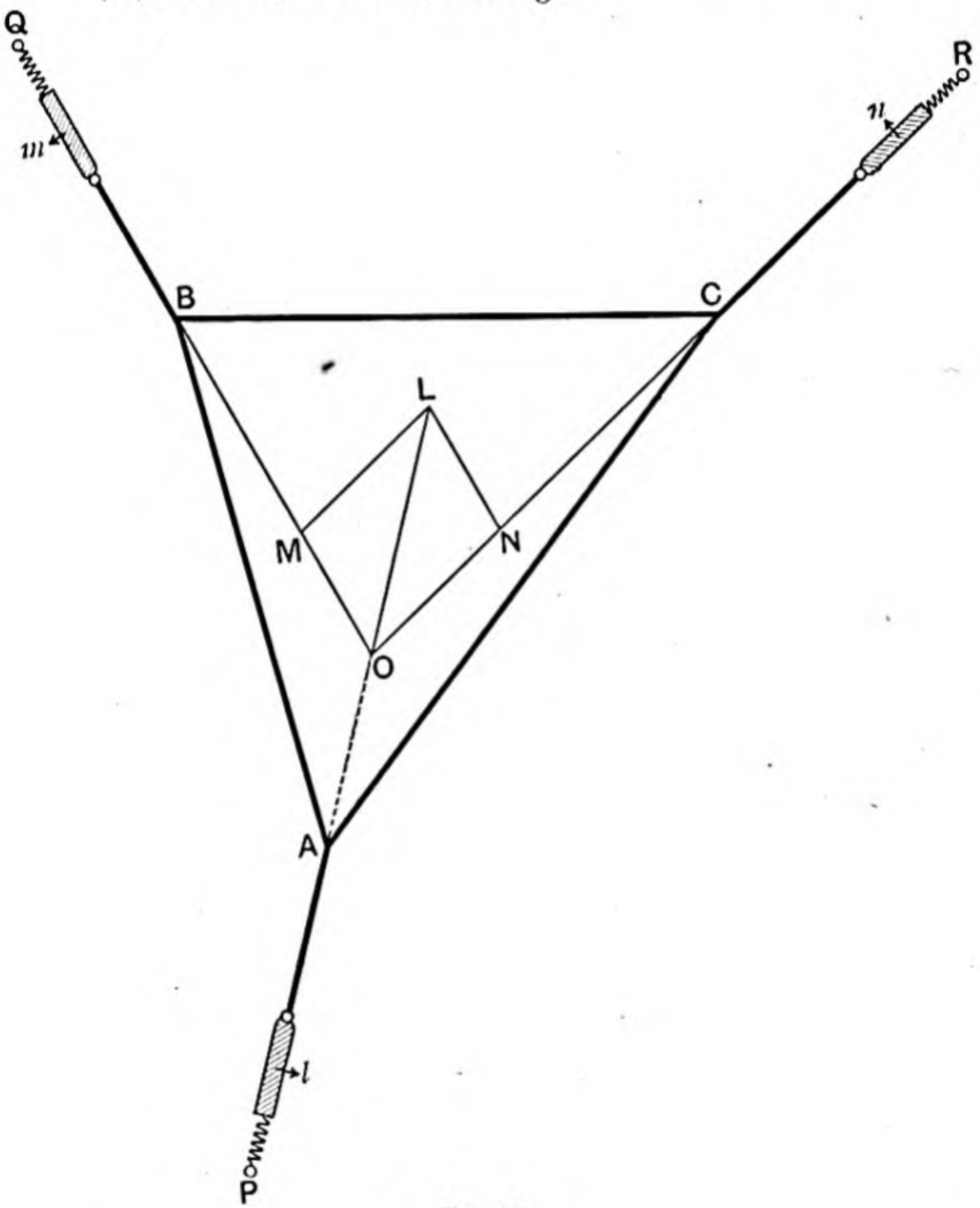

FIG. 10.

treated as a rigid body (p. 2)

Now consider the equilibrium of the string ABC. The forces which act externally upon it are the pulls at A, B, C. Of these three forces, the pulls at B and C act along, and are represented by, the lines OM and ON respectively. The

experiment shows us that the pull at A acts along, and is represented by, the line LO, so that the pull at A could be balanced by a force acting along, and represented by the line OL.

Thus it appears that the force acting along, and represented by, the line OL is equivalent to the two forces which act along, and are represented by, the lines OM and ON.

26. ~~Extension~~ Application of the Parallelogram of Forces.—In Art. 24 the line BC is equal and parallel to OD, and therefore represents the force Q in magnitude and direction. It is not necessary to draw the whole parallelogram in determining the resultant, for we can draw OB to represent the force P, then BC to represent the force Q, and OC will represent the resultant. Also, to avoid confusion in the figure, the forces can be represented to scale in a separate diagram. Thus,

To find graphically the resultant of two given forces.

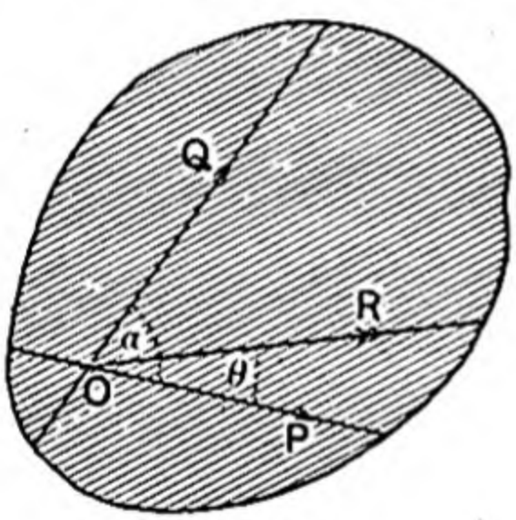

FIG. 11.

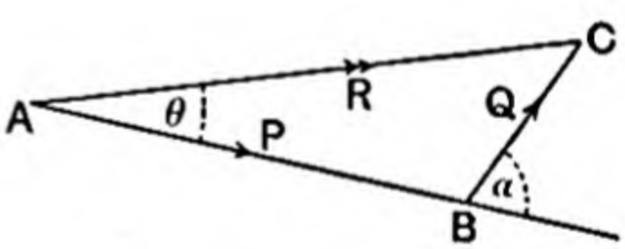

FIG. 11a.

Let the two given forces P and Q act along lines which intersect at O. With any suitable scale, draw AB equal to P units of length in the direction of the force P, and from B draw BC equal to Q units of length in the direction of the force Q. Then the straight line from A, where we started, to C, where we finished, represents the resultant in magnitude and direction. Suppose that AC contains R units of length. Then the resultant of the two forces P and Q is a force R acting along a line drawn through O parallel to AC.

This is the fundamental method for determining the resultant of two given forces, and from it can be deduced all the properties of the resultant of two forces which act along intersecting lines.

It is to be particularly noticed that the angle ABC is the *supplement* of the angle between the directions of the two forces P and Q.

Example.—Suppose it is required to determine the resultant of two forces of 8 and 3 pounds' weight acting at an angle of 120°.

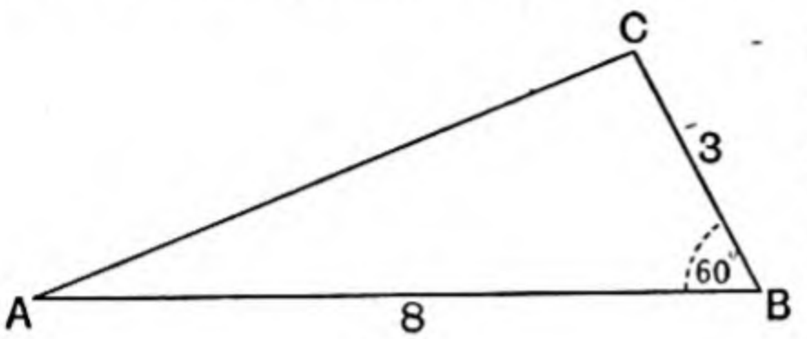

FIG. 12.

With any suitable scale make AB of length 8 units; make angle ABC = supplement of $120° = 60°$; and make BC of length 3 units. Then, on measurement, AC is found to be of length 7 units, and the angle BAC of magnitude $21\frac{3}{4}°$. Hence the resultant is 7 pounds' weight, making an angle of $21\frac{3}{4}°$ with the direction of the first force.

27. Calculation of the Resultant of two Forces.—In the preceding article, let α be the angle between the forces. Then, in the triangle ABC, the angle $ABC = \pi - \alpha$, while the sides that include this angle are of lengths P units and Q units. Hence, as the base contains R units of length, we have, by trigonometrical formula,

$$R^2 = P^2 + Q^2 - 2PQ \cos(\pi - \alpha)$$

$$\therefore R^2 = P^2 + Q^2 + 2PQ \cos \alpha \quad . \quad . \quad (i)$$

Let θ be the angle that the resultant makes with the force P.

Then $$\widehat{BAC} = \theta \text{ and } \widehat{ACB} = \alpha - \theta$$

Hence, as the sides of a triangle are proportional to the sines of the opposite angles,

$$P \sin \theta = Q \sin(\alpha - \theta)$$

$$= Q \sin \alpha \cos \theta - Q \cos \alpha \sin \theta$$

$$\therefore (P + Q \cos \alpha) \sin \theta = Q \sin \alpha \cos \theta$$

$$\therefore \tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha} \quad . \quad (ii)$$

The equations (i) and (ii) determine the magnitude and direction of the resultant.

Example.—The resultant of 8 and 3 pounds' weight acting at an angle of 120° is

$$\sqrt{64 + 9 + 2 \cdot 8 \cdot 3 \cdot \cos 120°} \text{ pounds' weight}$$

$$= \sqrt{64 + 9 - 24} \text{ pounds' weight}$$

$$= 7 \text{ pounds' weight;}$$

and the angle that it makes with the 8 pounds' weight

$$= \tan^{-1} \frac{3 \sin 120^\circ}{8 + 3 \cos 120^\circ}$$
$$= \tan^{-1} \frac{3\sqrt{3}}{16 - 3}$$
$$= \tan^{-1} \frac{3}{13}\sqrt{3}$$
$$= \tan^{-1} \cdot 3997 \ldots$$
$$= 21\tfrac{3}{4}^\circ \text{ nearly (by mathematical tables).}$$

28. In the preceding article, if P and Q are equal, the triangle ABC becomes isosceles.

In this case R bisects the angle between P and Q.

Also, the perpendicular from B upon AC divides AC into two parts each of length $P \cos \frac{\alpha}{2}$.

Thus, the resultant of two equal forces P acting at an angle α is of magnitude $2P \cos \frac{\alpha}{2}$, and acts along a line bisecting the angle α.

Example.—The resultant of two forces each of 10 pounds' weight acting at an angle of 60° is

$$2 \cdot 10 \cdot \cos 30^\circ \text{ pounds' weight}$$
$$= 10\sqrt{3} \text{ pounds' weight}$$
$$= 17 \cdot 32 \text{ pounds' weight.}$$

EXAMPLES II

1. If the side of a square denotes a force of 4 pounds' weight when one foot represents a weight of 3 pounds, what will the diagonal denote when one inch represents a weight of 6 ounces?

2. $OACB$ is a rectangle, the sides OA and OB being of lengths 16 and 12 inches respectively. Find the resultant of 4 pounds' weight acting along OA and 3 pounds' weight acting along OB.

Find also the resultant of 3 pounds' weight acting along OA and 4 pounds' weight acting along OB.

3. Show that the resultant of forces P and Q, acting at right angles, is of magnitude $\sqrt{P^2 + Q^2}$, and makes an angle $\tan^{-1} \frac{Q}{P}$ with the first force.

4. Find the magnitude of the resultant of the following pairs of forces acting at right angles.

(i) 7 and 24 pounds' weight.
(ii) 10 and 24 pounds' weight.
(iii) 18 and 24 pounds' weight,
(iv) 45 and 24 pounds' weight.

(v) 70 and 24 pounds' weight.
(vi) $94\frac{1}{2}$ and 24 pounds' weight.
(vii) 25·2 and 24 pounds' weight.
(viii) 43 and 21 pounds' weight.

and in each case give the tangent of the angle that the resultant makes with the larger force.

5. The resultant of two forces at right angles is of magnitude 65 pounds' weight. If one of the forces is of given magnitude, determine the magnitude of the other.

(i) When the given force is 16 pounds' weight.
(ii) ,, ,, 52 ,, ,,
(iii) ,, ,, 60 ,, ,,
(iv) ,, ,, 62·4 ,, ,,

6. The resultant of two forces P and Q is in a direction at right angles to the force P. Show that the cosine of the angle between the forces is $-\frac{P}{Q}$.

7. Find the magnitude of the resultant of two forces each of 10 pounds' weight.

(i) When they act at an angle of 120°
(ii) ,, ,, 135°
(iii) ,, ,, 150°

8. Find the resultant of

(i) 8 and 5 pounds' weight acting at an angle of 120°
(ii) 5 ,, 3 ,, ,, ,, ,, 60°
(iii) 8 ,, 7 ,, ,, ,, ,, 60°
(iv) 15 ,, 7 ,, ,, ,, ,, 120°
(v) 15 ,, 8 ,, ,, ,, ,, 120°
(vi) 6 ,, 7 ,, ,, ,, ,, 45°

and in each case give the cosine of the angle that the resultant makes with the larger force.

9. At what angle must forces of 8 and 3 pounds' weight be inclined, in order that their resultant may be 7 pounds' weight?

10. Find the cosine of the angle at which forces of 10 and 17 pounds' weight must be inclined, in order that their resultant may be 21 pounds' weight.

11. The resultant of two forces, acting at right angles, is 13 pounds' weight, and the difference between them is 7 pounds' weight. Find the forces.

12. The resultant of two forces of given magnitudes is greatest when the forces act in the same direction, and least when they act in opposite directions.

13. The resultant of two forces P and Q is of magnitude R. If one of the two forces is reversed in direction, the resultant of the two becomes R'. Prove that $R^2+R'^2=2(P^2+Q^2)$.

14. The resultant of forces of 11 and 7 pounds' weight is 14 pounds' weight. Find what the magnitude of the resultant becomes when one of the two forces is reversed in direction.

15. The resultant of two forces, one of which is double of the other, is 13 pounds' weight. If one of them is reversed, the other remaining unaltered, the resultant becomes 9 pounds' weight. Find the magni-

tudes of the forces and the cosine of the angle at which they must be inclined in the first case.

16. ABC is an equilateral triangle and D is the middle point of AB; if AR represents the resultant of forces represented by AD and AC, show that $AR = \sqrt{7}$ times AD.

17. The resultant of two equal forces P, acting at an angle of 60°, is of magnitude R. Show that the resultant of P and $\frac{1}{3}R$, acting at right angles, is of magnitude $\frac{2}{3}R$.

18. Two forces P and Q are inclined at an angle θ. If Q is replaced by a force $P+Q$, acting in the opposite direction, it is found that the resultant remains of the same magnitude as before. Determine the value of θ.

19. The resultant of two forces is 17 pounds' weight when they act at right angles, and 13 pounds' weight when they act at an angle of 120°. Find the forces.

20. The resultant of two equal forces P, acting at an angle θ, is of magnitude R, and the resultant of P and $\frac{1}{3}R$, acting at right angles, is of magnitude $\frac{2}{3}R$. Find the value of θ.

21. The resultant R of two forces P and Q is equal in magnitude to Q. Show that Q and R make supplementary angles with P.

22. AOA' and BOB' are two straight lines intersecting at right angles. Find the magnitude of the resultant of forces of 98, 91, 30, 40 pounds' weight acting along OA, OB, OA', OB' respectively, and give the tangent of the angle that it makes with OA.

23. The resultant R of two forces P and Q, acting at any given angle, makes an angle ϕ with the direction of the force Q. Show that the resultant of P and $(Q+R)$, acting at the same angle, is of magnitude $2R\cos\frac{1}{2}\phi$ and makes an angle $\frac{1}{2}\phi$ with the direction of $(Q+R)$.

24. Two forces P and Q act at an angle of 120°. Show that if the greater force P is replaced by a force $(P-Q)$, then the magnitude of the resultant is unaltered. Prove also that the sum of the angles which the resultant makes with the force P in the two different cases is 60°.

25. Two forces P and Q, acting along intersecting lines, have a resultant R. If the force Q is reversed in direction and changed into $\dfrac{R^2-P^2}{Q}$, show that the resultant is still of magnitude R.

29. Property of the Line of Action of the Resultant of Two Forces.—*From a point L perpendiculars, of lengths p and q, are drawn upon the lines of action of two forces P and Q respectively, which act along intersecting lines. It is required to prove that* (i) *If L is in the line of action of the resultant of P and Q, then $Pp = Qq$;* (ii) *If L is in the angle between the two forces P and Q (or in the vertically-opposite angle) and $Pp = Qq$, then L is a point in the line of action of the resultant of the two forces.*

Let the two forces P and Q, represented by AB and BC respectively, act along lines which intersect at O; and let the

circle on OL as diameter meet the lines of action of P and Q in H and K respectively. Then LH and LK are the perpendiculars, and are of lengths p and q respectively.

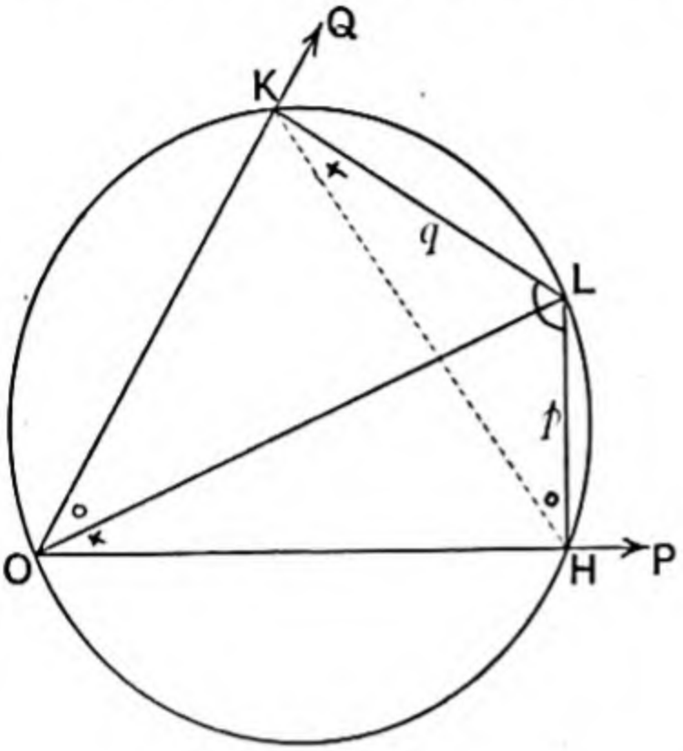

FIG. 13.

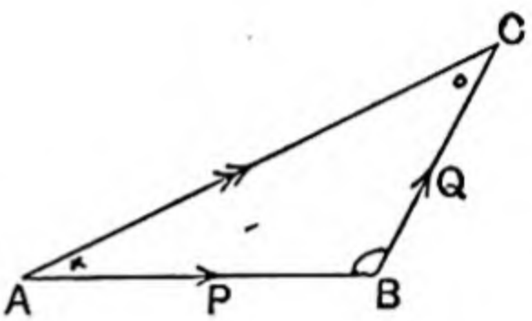

FIG. 13a.

(i) If L is in the line of action of the resultant of P and Q, the line OL is parallel to AC.

$$\therefore \widehat{CAB} = \widehat{LOH} = \widehat{LKH}.$$

Also $\widehat{ABC}$ = supplement of $\widehat{KOH} = \widehat{KLH}$.

$\therefore$ the triangles ABC and KLH are similar.

Thus $P : Q = AB : BC = KL : LH = q : p$.

$$\therefore Pp = Qq.$$

(ii) If L is in the angle between the two forces P and Q (or in the vertically opposite angle), we have

$\widehat{KLH}$ = supplement of the angle between the forces = $\widehat{ABC}$.

Also, if $Pp = Qq$, we have

$$KL : LH = q : p = P : Q = AB : BC,$$

$\therefore$ the triangles KLH and ABC are similar.

$$\therefore \widehat{BAC} = \widehat{LKH} = \widehat{LOH}$$

$\therefore$ OL is parallel to AC.

Thus L is in the line of action of the resultant of the forces P and Q.

Example.—Suppose it is required to determine where the line of

action of the resultant of the two known forces P and Q, indicated in Fig. 14, intersects the given line AB.

Let c be the known length of AB, and let the line of action of the resultant intersect AB at the point L at a distance x from A. Then the perpendiculars from L upon the lines of action of P and Q are $x \sin \alpha$ and $(c-x) \sin \beta$ respectively. Hence x can be found from the equation

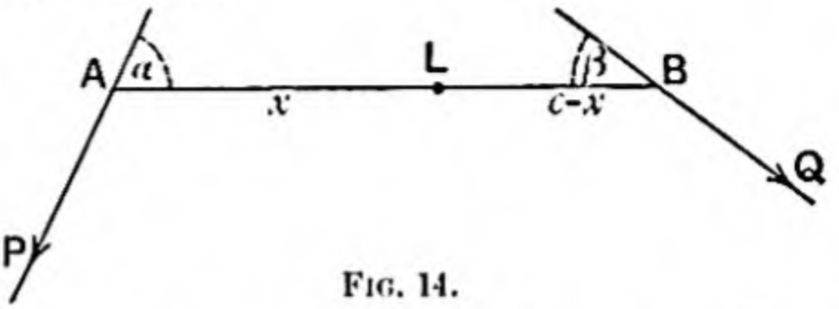

FIG. 14.

$$P\, x \sin \alpha = Q\,(c-x) \sin \beta.$$

If the direction of the force Q be reversed, the point L will lie in BA produced. In this case, if L is at a distance x' from A, we have

$$P\, x' \sin \alpha = Q\,(c+x') \sin \beta.$$

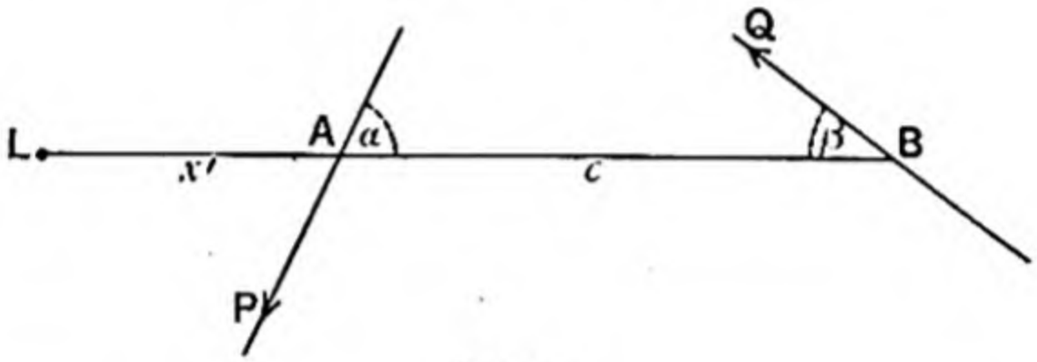

FIG. 15.

The result is the same as we should have obtained from the equation above by changing the sign of Q and writing $-x'$ instead of x.

***30. Resultant of λ (OB) and μ (OD).** — *If two forces, acting along OB and OD, are represented in magnitude and direction by λ times OB and μ times OD respectively, and if a point E is taken in BD such that λ times $BE = \mu$ times ED, then the resultant acts along OE and is represented in magnitude and direction by $(\lambda + \mu)$ times OE.*

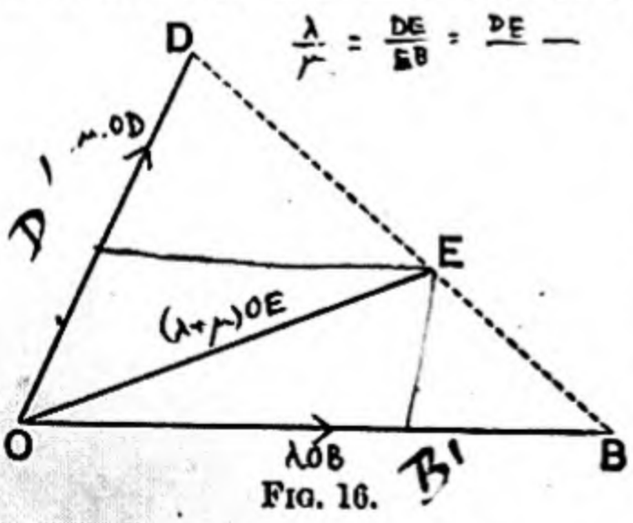

FIG. 16.

The force which acts along OB is equivalent to two forces represented by λ times OE and λ times EB, *both acting through O.*

The force which acts along OD is equivalent to two forces represented by μ times OE and μ times ED, *both acting through O.*

Replacing the two given forces by these pairs of components,

we see that the two forces represented by λ times EB and μ times ED, being equal and opposite and acting along the same straight line through O may be removed.

Thus the resultant acts along OE and is represented by $(\lambda + \mu)$ times OE.

In particular, if the forces are represented by OB and OD, and if E is the middle point of BD, then the resultant acts along OE and is represented by twice OE.

31. Resolution of a Force into Two Components.—*To resolve a given force into two components in two given directions.*

Let OC be the line of action of the given force P, and let OH and OK be straight lines drawn through O in the given directions.

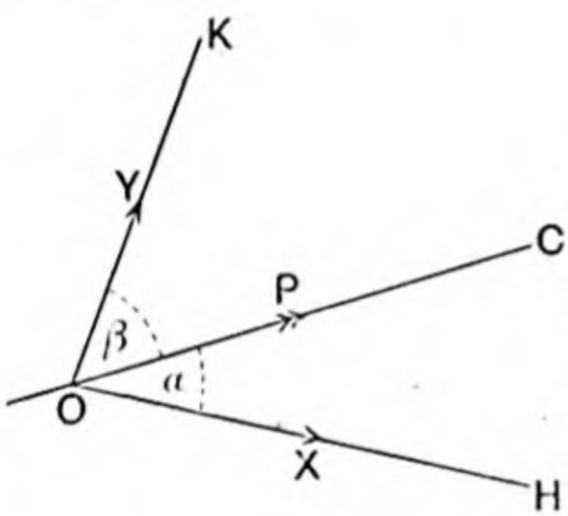

FIG. 17.

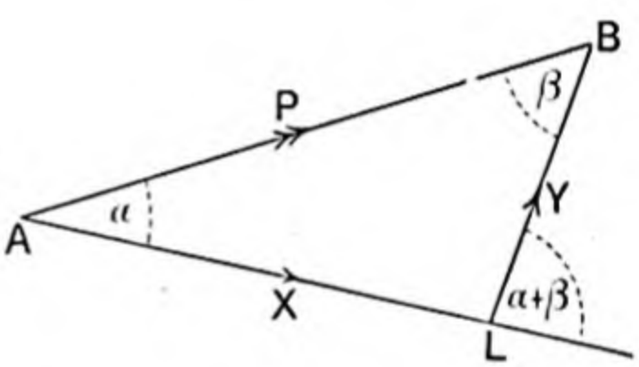

FIG. 17a.

(i) *By Geometrical Construction.*—Taking AB in the direction of the force P and of length P units, draw AL parallel to OH and BL parallel to KO, meeting in L.

Then if AL and LB contain X and Y units of length respectively, the given force is equivalent to two forces X and Y acting along OH and OK respectively. Thus we determine the required components by measuring AL and LB.

(ii) *By Calculation.*—Let the angles HOC and COK be α and β respectively. Then in the triangle ABL, the angles A and B are α and β respectively, and the external angle at L is $\alpha + \beta$; also the side AB is of known length P. Thus, to find X and Y, we have

$$\frac{X}{\sin\beta} = \frac{Y}{\sin\alpha} = \frac{P}{\sin(\alpha+\beta)},$$

cf. Art. 42.

$$\therefore \begin{cases} X = P \sin\beta \operatorname{cosec}(\alpha+\beta) \\ Y = P \sin\alpha \operatorname{cosec}(\alpha+\beta). \end{cases}$$

If α and β are equal, then the triangle ABL becomes isosceles, so that the forces X and Y are equal.

In this case the perpendicular from L upon AB divides AB into two parts each of length $X \cos \alpha$. Thus X can be determined from the equation.

$$2X \cos \alpha = P.$$

32. The Resolved Part of a Force in a Given Direction.—In the preceding article, suppose that the direction of OH is given, while the direction of OK is unknown. Then the position of the line AL is known, but not the position of the point L. Thus X has different values according to the direction in which OK is drawn.

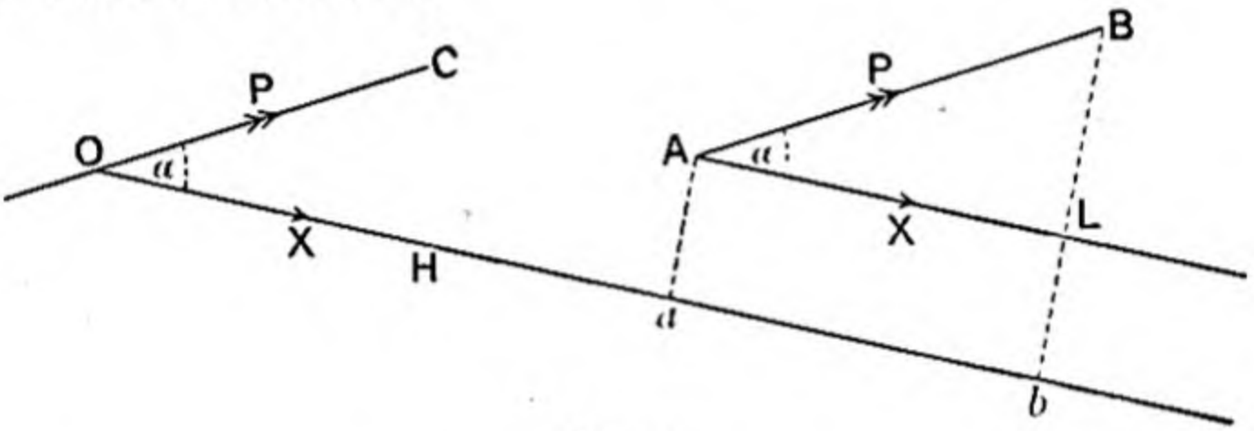

FIG. 18.

Of the different values of X we will choose that particular one which requires that the remaining component Y should be in a direction perpendicular to OH. In this case we draw BL perpendicular to AL, and thus determine the position of the point L.

Def.—*That component of a given force in a given direction which requires the remaining component to act at right angles to the given direction, in order that the two components may be together equivalent to the given force, is called the resolved part of the given force in the given direction.*

Draw Aa and Bb perpendiculars upon OH. Then, evidently, $ab = AL$.

Thus the resolved part of a given force in a given direction is represented by the *projection* upon the given direction of the straight line which represents the given force.

When the projection of AB upon OH is negative (as it will be when α is obtuse) the resolved part X is also negative.

Again, we have $ab = AL = AB \cos \alpha$

$$\therefore X = P \cos \alpha$$

And X is positive or negative according as $\cos \alpha$ is positive or negative.

Thus the resolved part of a given force in a given direction is given in magnitude and sign by the product of the measure of the given force and the cosine of the angle it makes with the given direction.

The resolved part of the force P in a direction at right angles to OH is of value $P \cos\left(\frac{\pi}{2} - \alpha\right) = P \sin \alpha$.

The resolved part of a force in its own direction is itself.

The resolved part of a force in a direction at right angles to its own direction is zero.

The student should carefully distinguish between the resolved part of a force in a given direction and a component of the force in that direction. In Fig. 17 the resolved parts of the force P in the directions OH and OK are of values $P \cos \alpha$ and $P \cos \beta$ respectively. These are not together equivalent to the given force P, except in the case where $\alpha + \beta = \frac{\pi}{2}$.

33. The Principle of Resolved Parts.—*The sum of the resolved parts, in any direction, of two forces which act along intersecting lines is equal to the resolved part of their resultant in that direction.*

In the figure of Art. 26 draw OX, any straight line through O, and suppose that perpendiculars are drawn from A, B, C, upon OX.

Then it is geometrically obvious that the sum of the projections of AB and BC upon OX is equal to the projection of AC upon OX, each projection having its own proper sign.

But the projection of any line represents the resolved part of the force represented by that line.

Thus the sum of the resolved parts of the forces P and Q is equal to the resolved part of their resultant R, each resolved part having its own proper sign.

If the forces are as indicated in Fig. 19, we have from the proposition proved above

$$R \cos \theta = -P \cos \alpha + Q \cos \beta.$$

This is described as *resolving along OX*.

In practice the value of the resolved part of the force P is obtained as follows:—Its resolved part estimated in the direction XO produced is $P \cos \alpha$; hence its resolved part in direction OX is $-P \cos \alpha$.

Resolving perpendicular to OX we obtain

$$R \sin \theta = P \sin \alpha - Q \sin \beta.$$

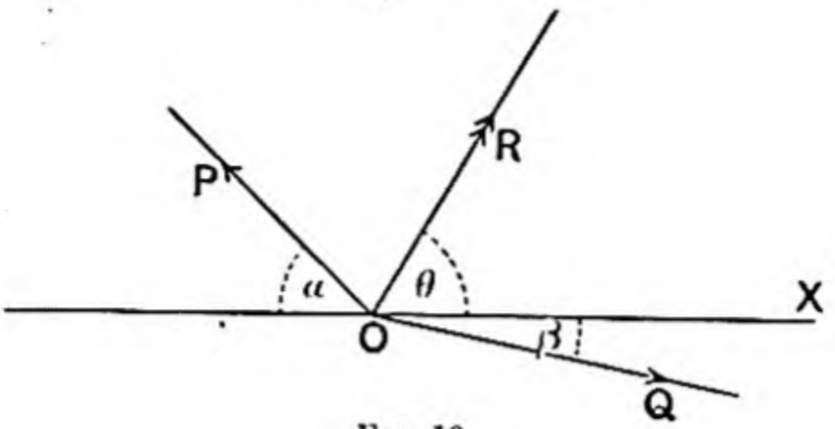

FIG. 19.

The two equations, obtained above, are sufficient to determine R and θ.

Squaring and adding, we have

$$R^2 = P^2 + Q^2 - 2PQ(\cos \alpha \cos \beta + \sin \alpha \sin \beta)$$
$$= P^2 + Q^2 - 2PQ \cos(\alpha - \beta)$$

Also,
$$\tan \theta = \frac{P \sin \alpha - Q \sin \beta}{-P \cos \alpha + Q \cos \beta}$$

The student is recommended to obtain the results of Art. 27 by this method, resolving in the direction of the force P and at right angles to this direction.

34. The following particular cases of the proposition proved above are important, and should be carefully noted :—

(i) The sum of the resolved parts of the two forces in the direction of their resultant is equal to the resultant.

(ii) The sum of the resolved parts of the two forces in a direction perpendicular to their resultant is zero.

Example.—Suppose it is required to find the value of the force X in order that the resultant of P and X may make angles α and β with the directions of P and X respectively, as indicated in Fig. 20.

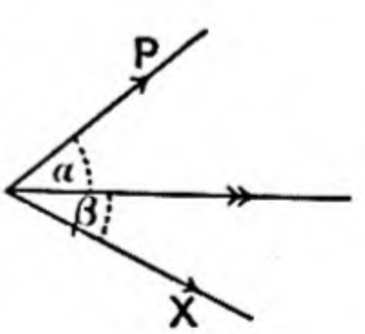

FIG. 20.

Resolving perpendicular to the direction of the resultant, we have

$$P \sin \alpha - X \sin \beta = 0$$
$$\therefore X = P \sin \alpha \operatorname{cosec} \beta.$$

Also, if X has this value, then, resolving in the direction of the resultant, we see that the magnitude of the resultant is

$$P \cos \alpha + X \cos \beta$$
$$= P \cos \alpha + \frac{P \sin \alpha \cos \beta}{\sin \beta}$$
$$= P \sin(\alpha + \beta) \operatorname{cosec} \beta.$$

EXAMPLES III

1. If the resolved part of a force in a given direction is half the force, what is the angle between the given direction and that of the force?

2. The cosine of the angle AOB is ·28, and BO is produced to C. OD and OE are the bisectors of the angles AOB and AOC respectively. A force of 100 pounds' weight acts along OA. Find the resolved parts of this force in directions OA, OB, OC, OD, OE.

Determine also the two components along OB and OE, which are together equivalent to the given force.

3. Resolve a force of 10 pounds' weight

(i) into two equal components inclined at an angle of 30°.
(ii) ,, ,, ,, ,, ,, ,, 135°.
(iii) ,, ,, ,, ,, ,, ,, 120°.
(iv) into two components making angles of 90° and 30° with it on opposite sides.
(v) into two components making angles of 75° and 45° with it on opposite sides.

4. Resolve a force of 12 pounds' weight into two components making angles of 60° and 15° with it on opposite sides.

What are the resolved parts of the given force in these two directions?

5. OA and OB are two straight lines inclined at an angle of 60°. Find the magnitude and direction of a force, acting at O, such that its *resolved part* along OA is 23 pounds' weight, and its *resolved part* along OB is 22 pounds' weight.

Find also the two *components*, along OA and OB, which are together equivalent to this force.

6. Two forces, one of which is 16 pounds' weight, are inclined at an angle of 120°. Find the magnitude of the other in order that the resultant may be at right angles to the given force. Determine also the magnitude of the resultant.

7. Resolve a given force P into two components, inclined at an angle of 120° so that one component may be double of the other.

8. Show how to resolve a given force into two components, one of which is of given magnitude and acts in a given direction.

9. A given force P is to be resolved into two components, one of which is of given magnitude Q in a given direction inclined at an angle α to the direction of P. Obtain formulæ giving the magnitude and direction of the other component.

10. Resolve a force of 15 pounds' weight into two components, one of which is 7 pounds' weight and makes an angle of 60° with the given force.

11. Find the cosine of the angle at which two forces of 40 and 25 pounds' weight must be inclined, in order that their resultant may make an angle $\cos^{-1}\frac{4}{5}$ with the greater force. Determine also the magnitude of the resultant.

12. Two forces are represented in magnitude and direction by AO and BO; show that their resultant is represented by twice CO, where C is the middle point of AB.

If the forces are represented by AO and OB, find their resultant.

13. D is the middle point of the side BC of a triangle ABC. Prove that the resultant of forces represented by AB and AC is double of the resultant of forces represented by AB and BD.

14. If D is the middle point of the base BC of a triangle ABC, and the resultant of forces represented by BA and BD is equal to the resultant of those represented by CA and CD, show that the triangle ABC is isosceles.

15. Two forces are represented in magnitude and direction by OA and $2OB$. Show that their resultant is represented by $3OC$, where C is one of the points of trisection of AB.

16. OA, OB, OR represent in magnitude, direction, and position two forces and their resultant. If OC and OD are two equal lines cut off from OA and OB respectively, and if OR meets CD in G, find the ratio of CG to GD.

17. AB is a straight line one foot in length. Forces of 5 and 2 pounds' weight act along AC and BD respectively, the angles BAC and ABD being 90° and 30° respectively. Find where the line of action of the resultant intersects AB or BA produced, (i) when AC and BD are on the same side of AB, and (ii) when AC and BD are on opposite sides of AB.

18. Forces P and Q act along the sides OA and OB respectively of the triangle OAB. Show that the line of action of their resultant divides AB in the ratio $Qa : Pb$, where a and b are the lengths of the lines OA and OB respectively.

19. Two forces act along OA and OB. Prove that (i) if the line of action of their resultant bisects AB, then the forces are proportional to OA and OB respectively; (ii) if the line of action of the resultant divides AB in the ratio $OA : OB$, then the forces are equal.

20. The sides OA, OB, AB of the triangle OAB are of lengths 5, 5, 6 inches respectively. Forces of 11 and 25 pounds' weight act along OA and OB respectively, and the line of action of their resultant meets AB at the point D. Find the length of AD, and show that it is equal to OD. Determine also the magnitude of the resultant.

21. The sides OA, OB, AB of the triangle OAB are of lengths 5, 5, 6 inches respectively. Forces of 7 and 3 pounds' weight act along OA and BO respectively, and the line of action of their resultant meets BA produced in D. Find the length of AD, and determine the magnitude of the resultant.

22. The sides OA, OB, AB of the triangle OAB are of lengths 21, 35, 28 inches respectively. Forces of 3 and 2 pounds' weight act along OA and OB respectively. Find the position of the point in AB through which the line of action of the resultant passes.

23. Forces P and Q act along the sides OA and OB respectively of the triangle OAB. Prove that (i) if $P : Q = \cos A : \cos B$, then the line of action of the resultant of P and Q passes through the centre of the circle which circumscribes the triangle; and (ii) if $P : Q = \cos B : \cos A$, then the resultant is perpendicular to AB.

***35. Formal Proof of the Parallelogram of Forces.**— We *assume* that the resultant of two *equal* forces acts along the bisector of the angle between the lines of action of its com-

ponents. For there is no reason why the resultant should make with one component an angle different to that which it makes with the other.

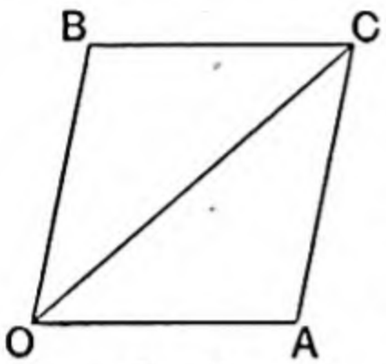

FIG. 21.

Now let two equal forces act along the lines *OA* and *OB*. Make *OA* equal to *OB*, and complete the rhombus *OACB*. Then, by elementary geometry, *OC* bisects the angle between *OA* and *OB*. Hence the resultant of the two forces acts along *OC*, and may be regarded as applied at the point *C*. There it may be resolved into two equal forces, of the same intensity as before, acting along *BC* and *AC*.

Thus, two equal forces acting along *OA* and *OB* are equivalent to two equal forces, of the same intensity, acting along *BC* and *AC*. But the last-mentioned pair (and therefore also the first-mentioned pair), could be balanced by two equal forces, of the same intensity as before, acting along *CB* and *CA*.

Thus, four equal forces, acting along the lines *OA*, *OB*, *CA*, *CB*, form a system in equilibrium.

Such a system of four forces will, for shortness of expression, be described as a *rhombus of forces*, and will be indicated in a diagram as in Fig. 22.

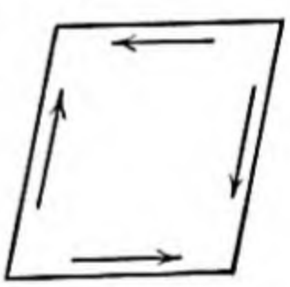
FIG. 22.

Now consider several such systems of forces, as indicated in Fig. 23, the whole figure forming a parallelogram *OACB*. Each rhombus of forces forms a system in equilibrium; therefore the whole system is in equilibrium.

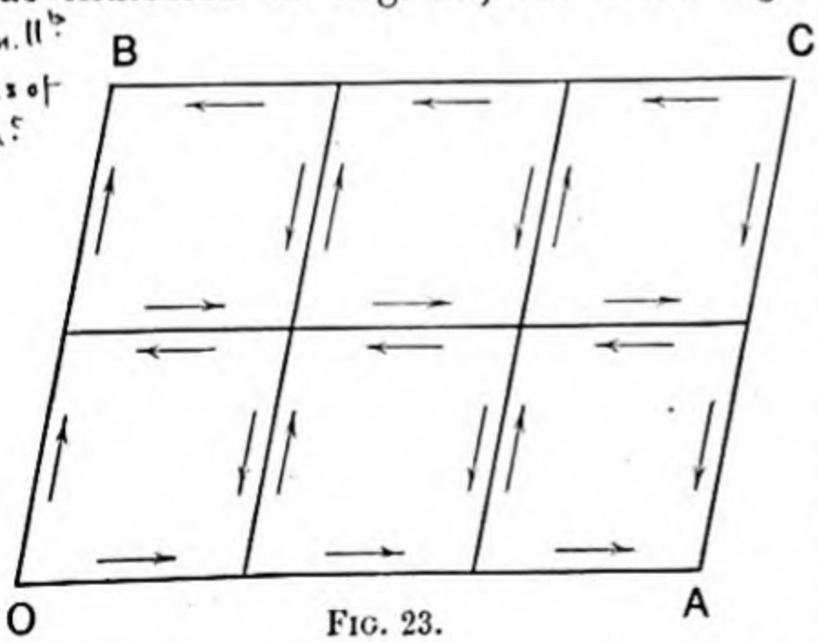

FIG. 23.

But all the forces which act along the lines *within* the parallelogram consist of pairs of equal and opposite forces acting along the same straight line, and can therefore be removed.

Hence the remaining forces, which act along the sides of the parallelogram, form a system in equilibrium.

But these remaining forces are equivalent to four forces acting along, and represented by *OA*, *OB*, *CA*, *CB*.

Hence, in any parallelogram *OACB*, *whose sides are commensurable*, four forces acting along, and represented by, *OA*, *OB*, *CA*, *CB* form a system in equilibrium.

We may remove the limitation that the sides of the parallelogram should be commensurable. For, if *OA* and *OB* were incommensurable, we could still find a fraction which differs from the true ratio of *OA* to *OB* by a quantity less than any assignable difference.

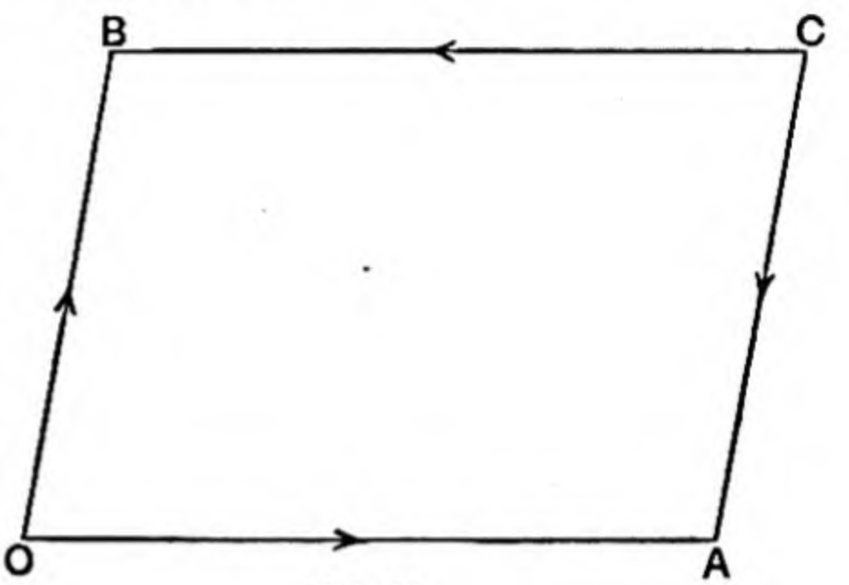

FIG. 24.

Thus we see that

In any parallelogram OACB, four forces acting along, and represented by, OA, OB, CA, CB form a system in equilibrium.

Now the first pair of forces are equivalent to a resultant acting through *O*, and the second pair to a resultant acting through *C*.

These resultants must be equal and opposite and act along the same straight line.

Therefore

If two forces act along, and are represented by, OA and OB, their resultant acts along OC.

We can now prove that *OC* represents the magnitude also of the resultant.

As the forces represented by *OA* and *OB* are equivalent to a force along *OC*, they can be balanced by an equal force acting along *CO*. Let such force be represented by *OD*, so that *COD* is a straight line. Complete the parallelogram *OBED* (see Fig. 25).

Then, by that which we have just proved, the forces represented by *OB* and *OD* can be replaced by some force along *OE*. Such force will balance the force represented by *OA*.

Hence *EOA* is a straight line.

Thus *EOCB* is a parallelogram.

$\therefore$ $OC = EB = DO$.

$\therefore$ OC and OD represent forces which are of equal magnitude.

But OD represents a force equal in magnitude to the resultant of the two forces represented by OA and OB.

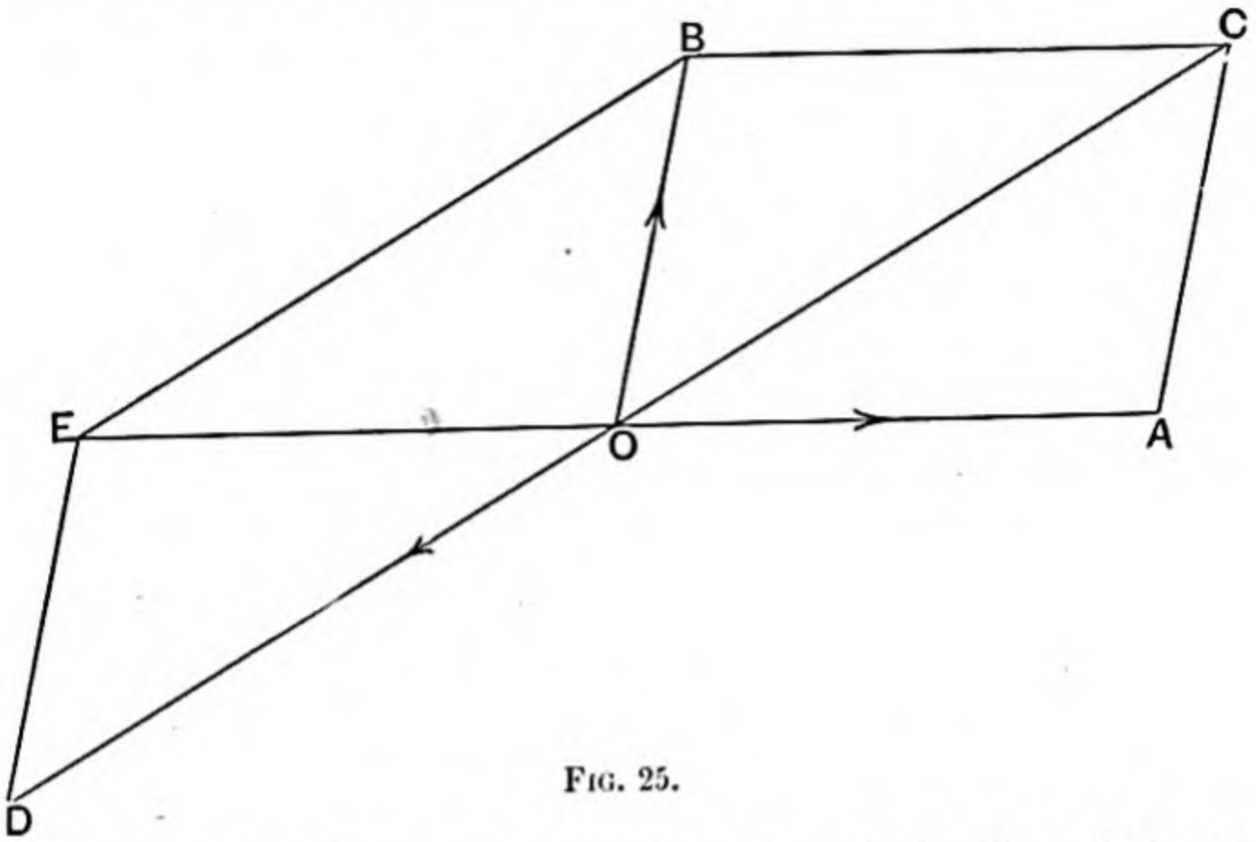

FIG. 25.

Therefore OC must represent the magnitude also of the resultant of the two forces which act along, and are represented by, OA and OB.

36. Ex. 1.—*Show how to resolve a given force P into two components, one of which is in a given direction inclined at an angle α to the given force, while the other is of given magnitude Q.*

(i) *By Geometrical Construction.*—Take AB to represent the given force P, and draw AH in the given direction of the first component, so that the angle HAB is of magnitude α.

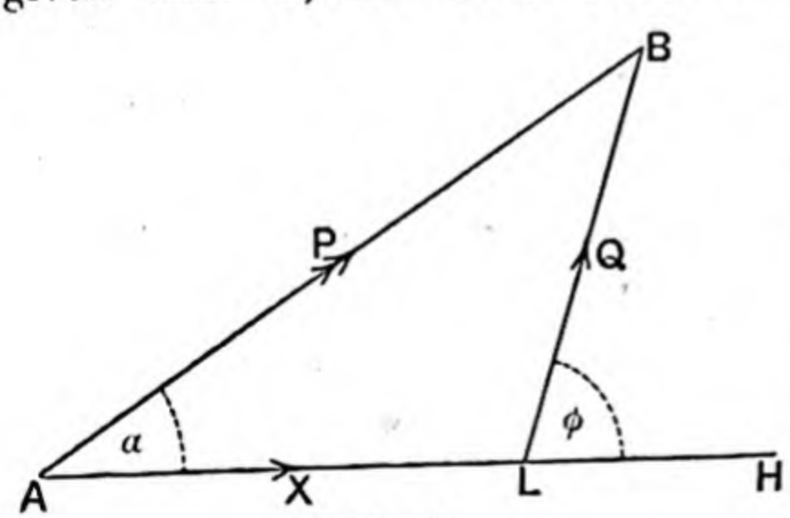

FIG. 26.

The two components will be represented by AL and LB, where AL lies along AH and LB is of magnitude Q. Thus, to get the position of L, we describe a circle with its centre at B and its radius of magnitude Q, and determine the points,

if any, in which the circle intersects AH. We see that the problem may admit of two solutions.

(ii) *By Calculation.* — Let X be the measure of the first component, and ϕ the angle at which the second component is inclined to AH. Then in the triangle ABL, the side AL is of magnitude X, and the exterior angle at L of magnitude ϕ.

Hence, by trigonometrical formula,

$$Q^2 = P^2 + X^2 - 2PX \cos \alpha.$$

This is a quadratic equation to find X.

Also,

$$\frac{\sin ALB}{P} = \frac{\sin \alpha}{Q},$$

$$\therefore \sin \phi = \frac{P}{Q} \sin \alpha.$$

This equation determines the angle ϕ, which may be either acute or obtuse, the two values being supplementary to each other.

37. Ex. 2.—*The resultant of two forces is perpendicular to one of them. Show that, if this force be doubled, the new resultant will be equal in magnitude to the other force.*

(i) *Geometrically.*—Let AB and BC represent the two forces. Then AC represents their resultant and must be perpendicular to AB.

If the first force be doubled, it will be represented by A_1B, where A is the middle point of A_1B, and the new resultant will be represented by A_1C.

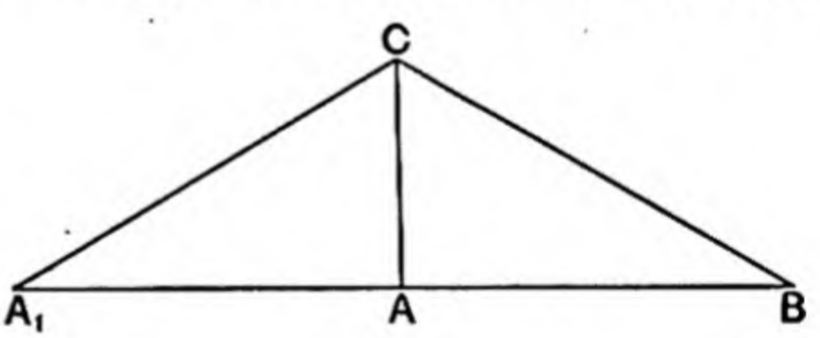

FIG. 27.

In the triangles, A_1AC and BAC, the sides A_1A and BA are equal, the side AC common, and the angles at A right angles; therefore these triangles are equal in all respects.

Thus $A_1C = BC$, so that the new resultant is equal to the force represented by BC.

(ii) *Analytically.* — Let the resultant of forces P and Q, acting at an angle α, be at right angles to the force P. Then the sum of the resolved parts of the two forces P and Q in the direction of the force P must be zero.

$$\therefore\ P + Q\cos\alpha = 0,$$

$$\text{so that}\qquad \cos\alpha = -\frac{P}{Q}.$$

Now let P be changed into $2P$; then the new resultant will be of magnitude

$$\sqrt{(2P)^2 + Q^2 + 2(2P)Q\cos\alpha},$$

$$= \sqrt{4P^2 + Q^2 + 4PQ\left(-\frac{P}{Q}\right)},$$

$$= Q.$$

*38. Ex. 3.—*Two forces are represented in magnitude and direction by A_1B_1 and A_2B_2. Show that their resultant is represented in magnitude and direction by twice AB, where A and B are the middle points of A_1A_2 and B_1B_2 respectively.*

Let the lines of action of the two forces intersect at a point O. Then the force represented by A_1B_1 is equivalent to two forces represented by A_1A and AB_1, *both acting through* O. Also, the force represented by A_2B_2 is equivalent to two forces represented by A_2A and AB_2, *both acting through* O,

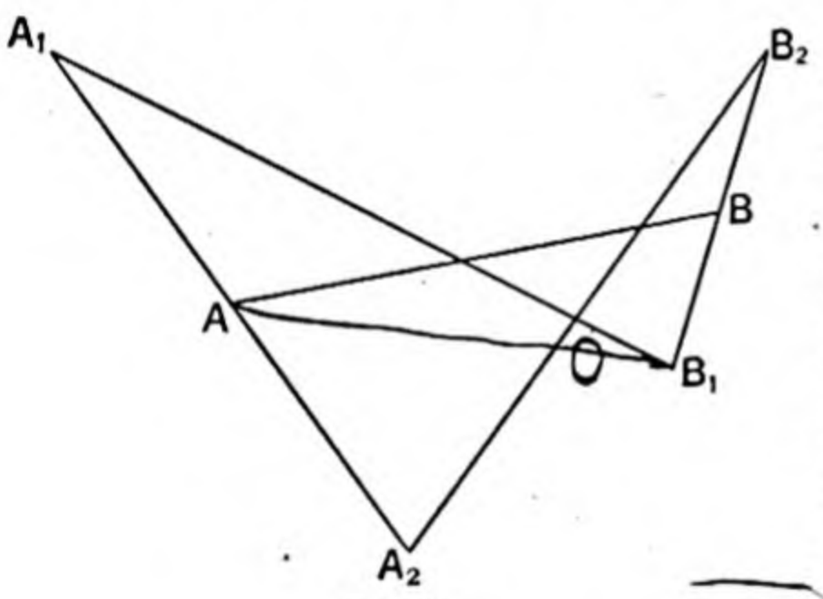

FIG. 28.

Replacing the two given forces by these pairs of components, we see that the two forces represented by A_1A and A_2A, being equal and opposite and acting along the same straight line through O, may be removed.

Thus the two given forces are equivalent to two forces represented by AB_1 and AB_2, *both acting through* O.

Hence the resultant is a force represented by twice AB, *acting along a line through* O.

*EXAMPLES IV

1. $ABCD$ is a given parallelogram, and P is any point in AD, or AD produced either way; two forces act along, and are represented

by, BP and PC. Show that, for all positions of P, the resultant is of fixed magnitude and acts along a fixed straight line.

2. The sides AB and DC of the quadrilateral $ABCD$ are parallel. Determine the magnitude, direction, and position of the resultant of two forces acting along and represented by AC and DB.

3. Two forces P and Q act along given intersecting lines; if P is increased by Q and Q by $\frac{1}{4}P$, the direction of the resultant is unaltered. Find the ratio of P to Q.

4. The resultant of two forces P and Q is at right angles to P; the resultant of P and Q', acting at the same angle, is at right angles to Q'. Prove that $QQ' = P^2$.

5. Two forces P and Q have a resultant R, and the resolved part of R in the direction of P is of magnitude Q. Show that the angle between the forces is $2\sin^{-1}\sqrt{\frac{P}{2Q}}$.

6. The resultant of two forces P and Q is of magnitude R. Show that if P be doubled, Q remaining unaltered, then the new resultant will be of magnitude $\sqrt{2P^2+2R^2-Q^2}$.

7. The resultant of two forces P and Q is of magnitude P. Show that if the force P be doubled, Q remaining unaltered, then the new resultant will be at right angles to Q.

8. Two forces P and Q act at an angle α. Prove that if Q be changed into Q', α and P remaining unaltered, the magnitude of the resultant will remain the same as before, provided $Q+Q' = -2P\cos\alpha$.

9. Two forces P and Q have a resultant R. Prove that if Q be changed into Q', the force P remaining unaltered, the magnitude of the resultant will remain the same as before, provided $QQ' = P^2 - R^2$.

10. If one of two equal forces be reversed and doubled, the other remaining unaltered, it is found that the magnitude of the resultant is unaltered. Find the original angle between the forces.

11. $ABCDEF$ is a regular hexagon. Find the resultant of forces P and $2P$ acting along AC and BF respectively.

12. Two forces, one of which is of given magnitude P, are inclined at a given angle α. Show how to determine the magnitude of the second force, in order that the resultant may be of given magnitude R.

13. Two forces, one of which is 35 pounds' weight, are inclined at an angle of 120°. Determine the magnitude of the second force, in order that the resultant may be 31 pounds' weight.

14. Two forces, one of which is of given magnitude P, are inclined at a given angle α. Show how to determine the magnitude of the second force, in order that the resultant may make a given angle β with the first.

15. One of two forces, which act at an angle 3α, is 8 pounds' weight, and the resultant makes an angle α with the given force. If $\cos\alpha = \frac{1}{8}$, find the other force and the magnitude of the resultant.

16. Show how to resolve a given force into two components of given magnitudes. Under what circumstances is the problem impossible of solution?

17. The resultant of 6 and 4 pounds' weight is 5 pounds' weight. Find the cosines of the angles that the resultant makes with its two components, and show that one angle is double of the other.

18. The line of action of the resultant of two forces P and Q divides the angle between them in the ratio 1 : 2. Show that the magnitude of the resultant is $\frac{P^2 - Q^2}{Q}$.

19. The resultant of two forces P and Q is of magnitude $\frac{P^2 - Q^2}{Q}$. Show that its line of action divides the angle between the forces P and Q in the ratio 1 : 2.

20. In the example of Art. 36, show that the product of the two values of X is $P^2 - Q^2$, and their difference is $2\sqrt{Q^2 - P^2 \sin^2 a}$.

21. A force of 8 pounds' weight is resolved into two components, one of which is 7 pounds' weight, while the other component is inclined at an angle of 60° with the given force. Determine the magnitude of the second force.

22. Show how to determine the angle at which two forces of given magnitudes P and Q must be inclined, in order that their resultant may make a given angle a with the force P.

23. The resultant of two forces of 8 and 7 pounds' weight is in a direction making an angle of 60° with the greater force. Determine (i) the magnitude of the resultant, and (ii) the angle between the forces, given $\sin 82° = \cdot 990$. . . .

24. The greatest resultant that two forces can have is of magnitude P, and the least is of magnitude Q. Show that, when they act at an angle a, their resultant is of magnitude $\sqrt{P^2 \cos^2 \frac{a}{2} + Q^2 \sin^2 \frac{a}{2}}$.

25. In the example of Art. 37, show that the new resultant will be inclined to the force $2P$ at an angle equal to the supplement of a.

26. Two forces, one of which is 17 pounds' weight, have a resultant of 30 pounds' weight. If the known force be reversed, the resultant becomes 40 pounds' weight. Find the magnitude of the other force.

27. Show how to resolve a given force into two components, such that their difference may be of given magnitude and the greater of them in a given direction.

28. Resolve a force of 7 pounds' weight into two components whose difference is 3 pounds' weight, the greater component being inclined at an angle $\cos^{-1} \frac{2}{3}$ with the given force.

29. Show how to resolve a given force into two components, such that their sum may be of given magnitude and one of them in a given direction.

30. Resolve a force of 35 pounds' weight into two components whose sum is 42 pounds' weight, one component being inclined at an angle of 60° with the given force.

31. Two forces are represented in magnitude and direction by the straight lines joining the middle points of opposite sides of a quadrilateral. Show that their resultant is represented in magnitude and direction by one of the diagonals of the quadrilateral.

32. The sides AB, AC of the triangle ABC are each of length 7 inches, and the base BC is of length 6 inches. Find the force which, together with a force of 7 pounds' weight acting along AB, will be equivalent to a force of 12 pounds' weight acting along CB.

33. The sides BC, CA, AB of the triangle ABC are respectively 17, 31, 40 inches long. Find the force which, together with a force of 4 pounds' weight acting along AB, will be equivalent to a force of $6\frac{1}{8}$ pounds' weight acting along AC.

34. The sides OA, OB, AB of the triangle OAB are of length 13, 15, 14 inches respectively. Find where the line of action of the resultant of the following pairs of forces intersects AB:—

(i) 39 pounds' weight along OA and 18 pounds' weight along OB
(ii) 39 ,, ,, OA ,, 25 ,, ,, OB
(iii) 39 ,, ,, OA ,, 45 ,, ,, OB
(iv) 39 ,, ,, OA ,, $81\frac{2}{5}$,, ,, OB
(v) 39 ,, ,, OA ,, 15 ,, ,, BO

35. Two forces of given magnitudes act in one plane at two given points of a rigid body. Show that, if their lines of action are turned round these points in the same direction through any two equal angles, then the magnitude of their resultant will remain unaltered, and its line of action will always pass through a fixed point.

[Note greater deflection of resultant: wh. is non-existant!]

Locus of O is a ⊙ thro' A, B.

Force □ is constant in shape + size ∴ $A\hat{O}R = A\hat{O}_1R_1$

∴ O_1R_1 passes thro' the same pt S on the ⊙ as OR.

CHAPTER III

EQUILIBRIUM OF THREE FORCES

39. The Triangle of Forces.—*If three forces, acting upon the same rigid body along concurrent lines, are represented in magnitude and direction by the sides of a triangle taken one way round, then the forces are in equilibrium.*

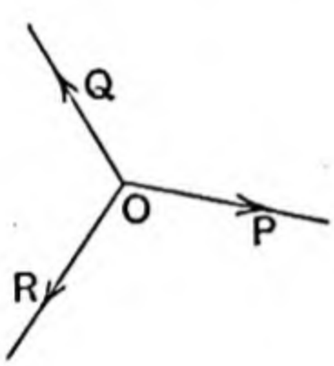

FIG. 29.

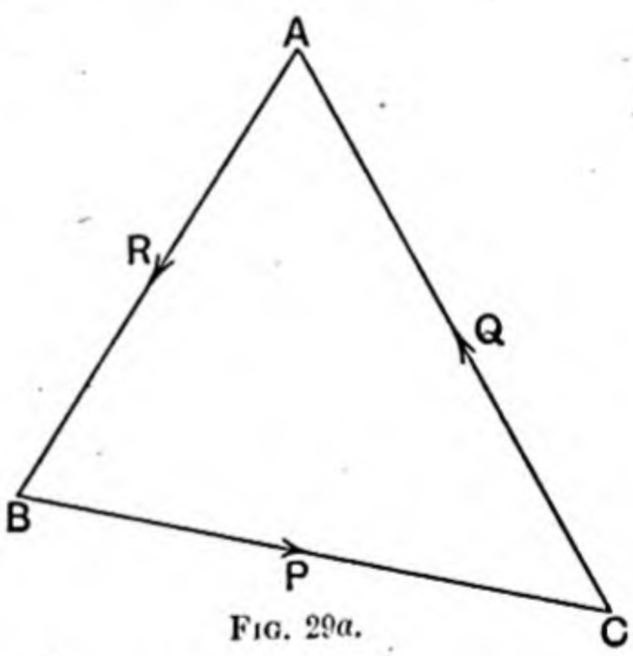

FIG. 29a.

Let three forces P, Q, R, acting along lines which intersect at O, be represented in magnitude and direction by the sides BC, CA, AB respectively of the triangle ABC.

Then, by the extension of the parallelogram of forces, the two forces P and Q may be replaced by a force represented by BA, acting along a line through O.

This reduces the system to two forces, namely, the resultant of P and Q represented by BA, and the force R represented by AB, both forces acting through the point O.

These two forces are of equal intensity and act in opposite directions along the same straight line through O.

Hence the system is in equilibrium.

40. Converse of the Triangle of Forces.—*If three forces, which are not all parallel, act upon a rigid body in one plane and keep it in equilibrium, then their lines of action are concurrent, and the forces can be represented in magnitude and direction by the sides of a triangle taken one way round.*

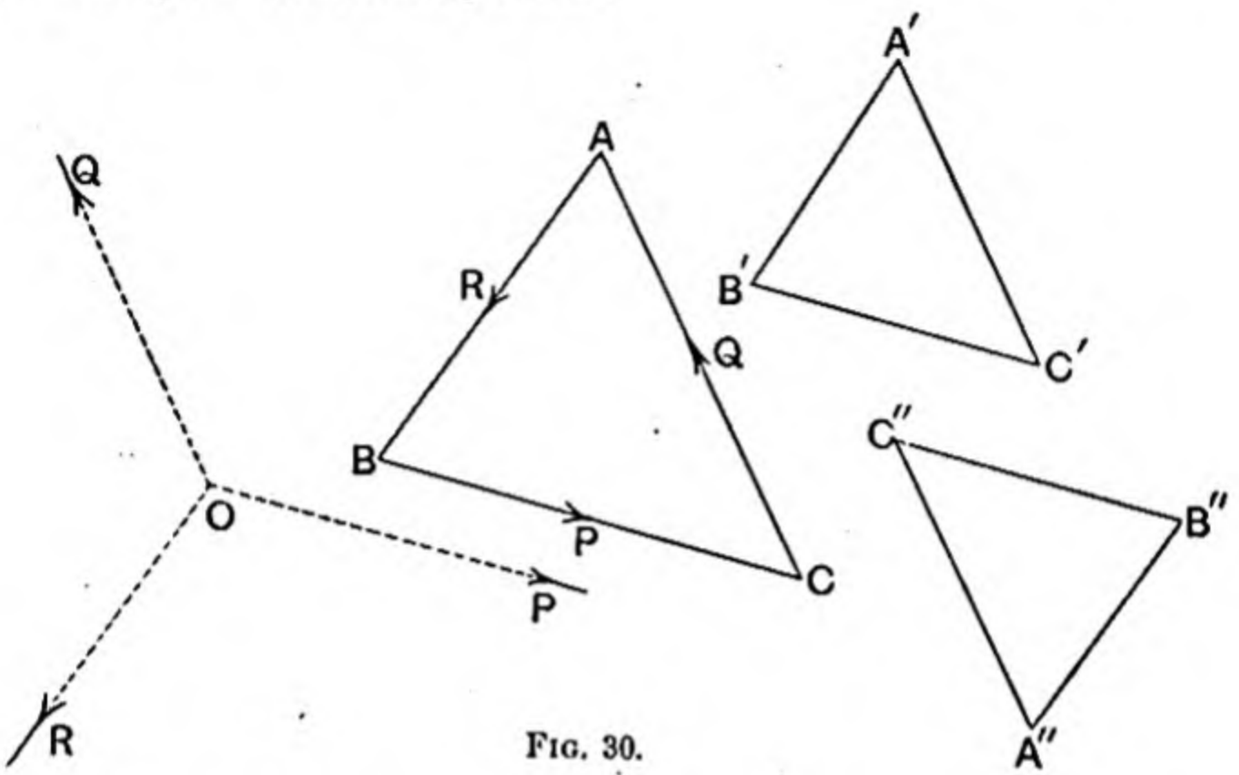

FIG. 30.

Let P, Q, R be the measures of three forces, which are not all parallel, and which act in one plane upon a rigid body and keep it in equilibrium.

As the forces are not all parallel, the lines of action of two of them, say P and Q, meet at a point O.

Let BC and CA be drawn in the directions of the forces P and Q, and of length P units and Q units respectively. Then, by the extension of the parallelogram of forces, the two forces P and Q may be replaced by a force represented by BA and acting along a line through O.

This reduces the system to two forces, namely, the resultant of P and Q represented by BA and acting through O, and the force R.

Now we are dealing with a system in equilibrium. Hence these two forces must be of equal intensity and act in opposite directions along the same straight line.

Therefore AB contains R units of length and is in the direction of the force R, and the line of action of the force R must pass through O.

Thus the lines of action of the forces must be concurrent, and

the force diagram is a triangle, whose sides taken one way round represent the three forces.

41. In the preceding article if any triangle, such as $A'B'C'$ or $A''B''C''$, is drawn with its sides parallel to the lines of action of the forces P, Q, R, it will be similar to the triangle ABC, and will therefore have its sides proportional to P, Q, R.

Thus we have a more general converse to the Triangle of Forces than that which we have enunciated and proved above, namely:

If three forces, which are not all parallel, act upon a rigid body in one plane and keep it in equilibrium, then their lines of action are concurrent; and if straight lines are drawn parallel to their lines of action so as to form a triangle, then the sides of the triangle are proportional to the forces to which they are respectively parallel.

As the lines of action of the forces meet at a point, the forces may be regarded as all acting at that point. In future, for shortness of expression, if the lines of action of a system of forces all pass through the same point, we shall describe the forces as *concurrent forces*, or as all *acting at a point.*

42. Lamé's Theorem.—*If three concurrent forces are in equilibrium, each is proportional to the sine of the angle between the other two.*

In the figure of Art. 40, the angles of the triangle ABC are respectively the supplements of the angles between the different pairs of forces, while the sides of the triangle are of lengths P, Q, R respectively.

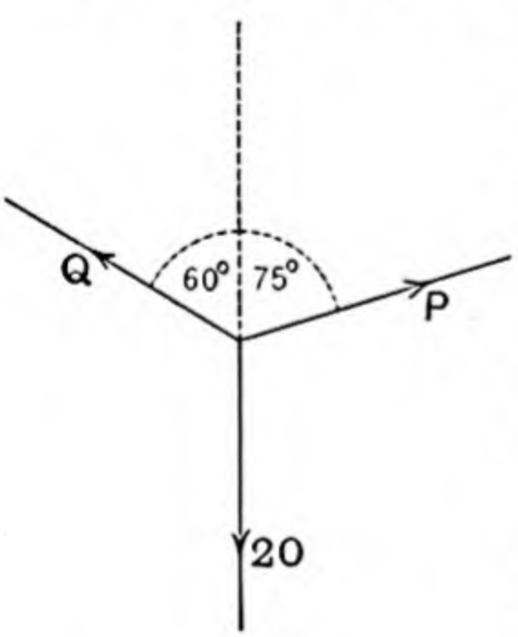

FIG. 31.

Now, by Trigonometry, the sides of a triangle are proportional to the sines of the angles opposite, and the sine of an angle is equal to the sine of its supplement.

Hence, denoting the angle between the forces Q and R by the notation $\widehat{(Q,R)}$, and similarly for the other pairs of forces, we have

$$\frac{P}{\sin\widehat{(Q,R)}} = \frac{Q}{\sin\widehat{(R,P)}} = \frac{R}{\sin\widehat{(P,Q)}}.$$

Example.—Suppose it is required to determine the values of P

and Q in order that the three forces indicated in Fig. 31 may be in equilibrium.

We have

$$\frac{P}{\sin(180^\circ - 60^\circ)} = \frac{Q}{\sin(180^\circ - 75^\circ)} = \frac{20}{\sin(60^\circ + 75^\circ)} = 20\sqrt{2},$$

$$\therefore \begin{cases} P = 20\sqrt{2}\sin 60^\circ = 10\sqrt{6} = 24{\cdot}49\ldots \\ Q = 20\sqrt{2}\sin 75^\circ = 10(\sqrt{3}+1) = 27{\cdot}32\ldots \end{cases}$$

43. Principle of Resolved Parts.—*If three concurrent forces are in equilibrium, the sum of their resolved parts in any direction is zero.*

In the figure of Art. 40 let OX be drawn in any direction through O, and suppose that perpendiculars are drawn from A, B, C, upon OX.

Then it is geometrically obvious that the sum of the projections of BC, CA, AB upon OX is zero, each projection having its own proper sign.

But the projection of any line represents the resolved part of the force represented by that line.

Hence the sum of the resolved parts of P, Q, R, in any direction is zero.

Ex. If the three forces indicated in Fig. 32 are in equilibrium, then, resolving along xX we have

$$P\cos\alpha = Q\cos\beta + R\cos\gamma.$$

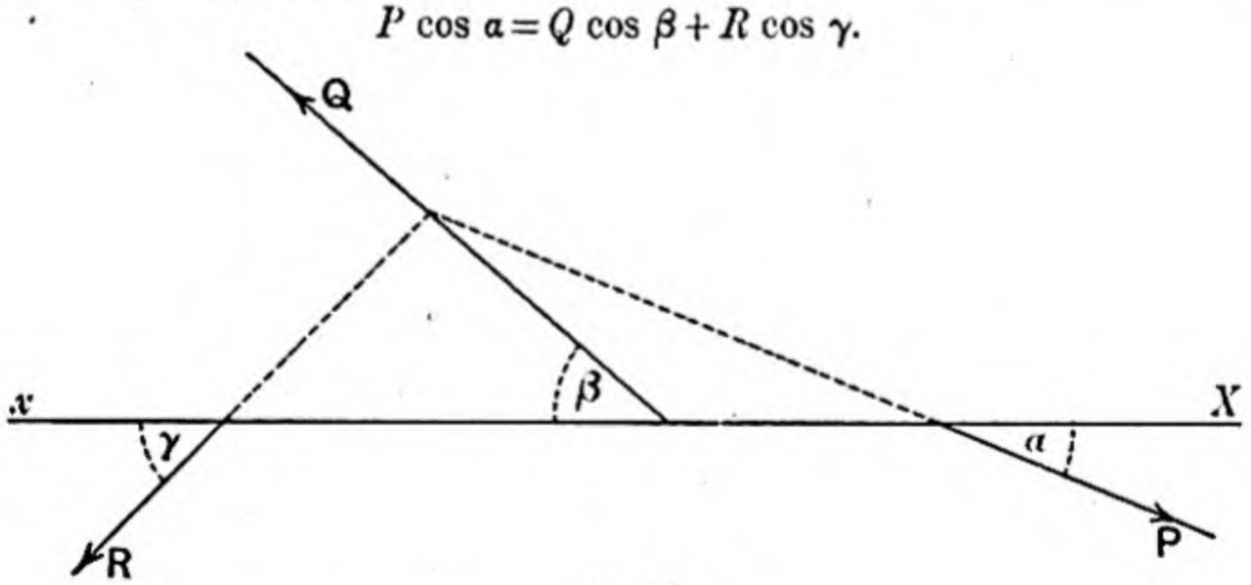

FIG. 32.

44. *If from any point in the line of action of one of three concurrent forces, which keep a body in equilibrium, perpendiculars are drawn upon the lines of action of the other two forces, then the perpendiculars are in the inverse ratio of the other two forces.*

This follows from Art. 29, since the line of action of any one force coincides with the line of action of the resultant of the other two.

45. Equilibrium of Three Strings.—Let three fine light strings OA, OB, OC, knotted together at O, be in equilibrium in a state of tension.

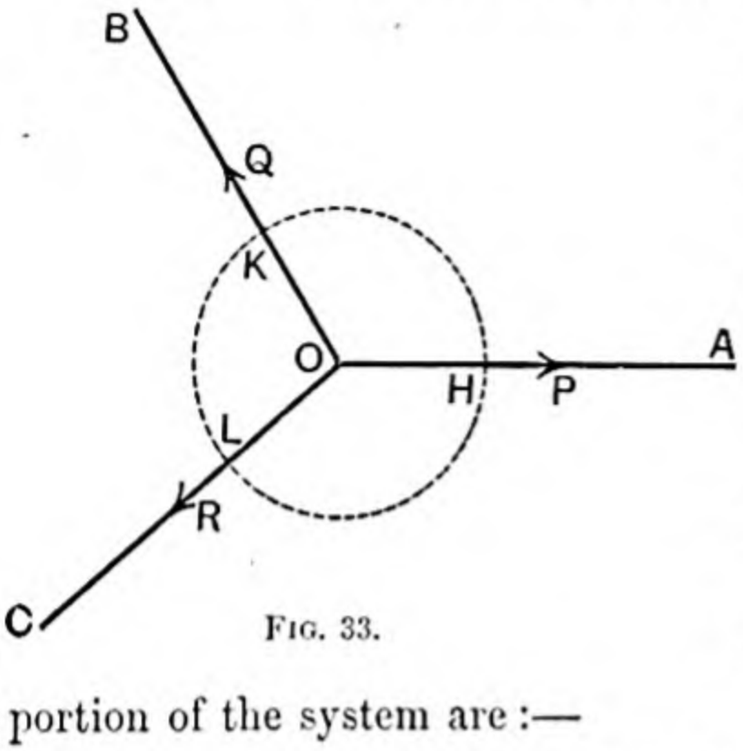

FIG. 33.

Let the tensions of the strings OA, OB, OC, be of magnitudes P, Q, R, respectively.

We have drawn a closed curve round O, enclosing the portions of string OH, OK, OL. The forces that act externally upon this portion of the system are :—

$$\begin{cases} \text{At } H \text{ a force } P \text{ in direction } HA \\ \text{At } K \text{ a force } Q \text{ in direction } KB \\ \text{At } L \text{ a force } R \text{ in direction } LC. \end{cases}$$

Thus the consideration of the equilibrium of the portion of matter in the neighbourhood of O comes under the subject of the present chapter, and the force diagram will be a triangle, in which the arrows go one way round.

This is briefly described as considering the equilibrium of O, and the force diagram is called the triangle of forces for the point O.

46. Ex. 4.—*Show how to arrange three forces of given magnitudes P, Q, R, so as to act at a given point O and produce equilibrium.*

(i) *By Geometrical Construction.*—Construct a triangle ABC so that the sides BC, CA, AB, may contain P, Q, R, units of length respectively ; and from O draw straight lines in the directions of BC, CA, AB (see Fig. 29). Then, if the forces P, Q, R are arranged to act along these straight lines respectively, we know by the Triangle of Forces that they will produce equilibrium.

The problem cannot be solved when any one of the forces is greater than the sum of the other two, as it is then impossible to construct the triangle.

When one of the forces, P, is equal to the sum of the other two, the point A falls in BC. Thus, in this case, Q and R must act in one and the same direction opposite to the force P.

(ii) *By Calculation.*—Since $\widehat{(Q,R)}$ = supplement of $\widehat{A}$ of the triangle ABC, we have

$$\cos \widehat{(Q,R)} = -\cos A = -\frac{Q^2+R^2-P^2}{2QR}.$$

This determines $\widehat{(Q,R)}$, and the angle between any other two forces can be found in the same way.

Numerical Example.—Suppose that forces of 7, 8, 5 pounds' weight are to be arranged so as to produce equilibrium, and that the angle between the forces of 8 and 5 pounds' weight is required. If the triangle is constructed with its sides of lengths 7, 8, 5, the angle between the last two sides is $\cos^{-1}\frac{8^2+5^2-7^2}{2\cdot 8\cdot 5}=\cos^{-1}\frac{1}{2}=60°$. The angle required must be the supplement of this, *i.e.* 120°

47. Ex. 5.—*Three forces act along given concurrent straight lines. The magnitude of one of the forces being known, it is required to determine the magnitudes of the other two, in order that the system may be in equilibrium.*

(i) *By Geometrical Construction.*—Let the three forces P, X, Y, in equilibrium, act along the three concurrent lines indicated in Fig. 34, P being known and X and Y unknown.

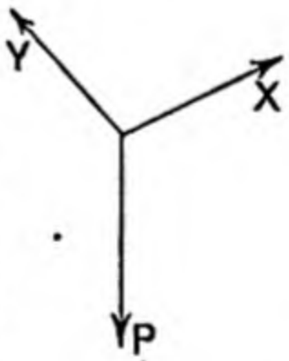

FIG. 34.

With any suitable scale draw AB to represent the known force P, and through B and A draw parallels to the lines of action of X and Y respectively, meeting in H. Then, by the converse of the triangle of forces, the sides of the triangle ABH, taken one way round, must represent the three forces. Hence, measuring BH and HA, we have the values of X and Y.

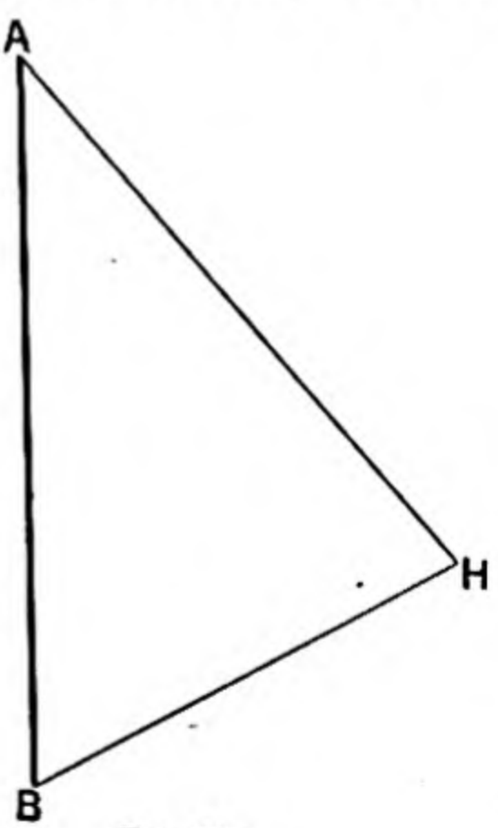

FIG. 34a.

(ii) *By Calculation*—(a) *First Method.*—If the lines of action of P, X, Y are given parallel to the sides of a known triangle

the values of X and Y can be at once determined, since the forces are proportional to the sides of the triangle.

Numerical Example.—Suppose that a mass of one cwt, suspended from a fixed point A by a fine light string AB, of length 20 inches, is pulled aside by a horizontal force until it rests at a distance of one foot from the vertical through A. Let it be required to determine the magnitude of the force applied, and the tension of the string.

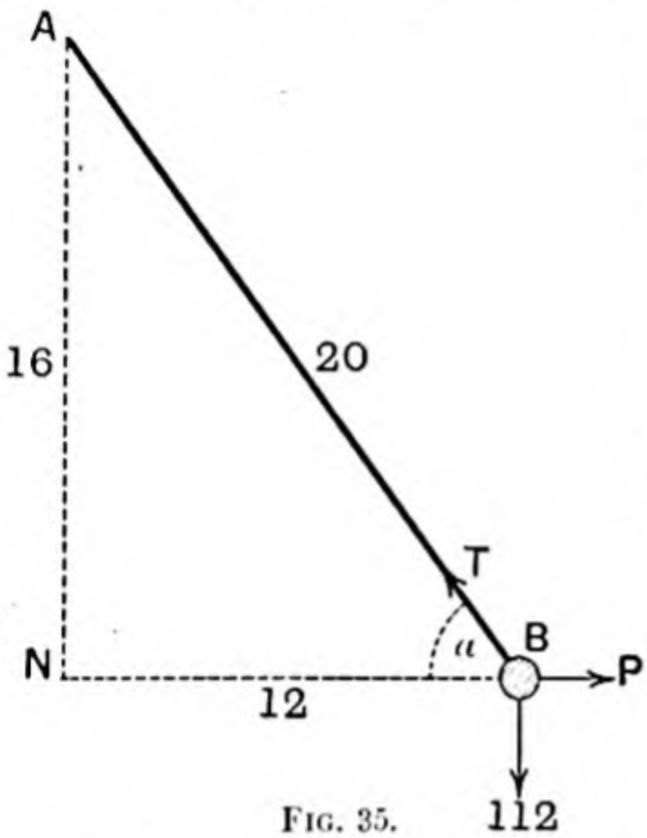

FIG. 35.

In the figure, NB is of length 12 inches.

Hence $AN = \sqrt{20^2 - 12^2}$ inches
$= 16$ inches.

The triangle ANB has its sides parallel to the three forces which act upon the mass. Hence, on the scale in which 16 inches represents 112 pounds' weight, 12 inches will represent the horizontal force applied, and 20 inches the tension of the string.

Thus the force applied $= 12 \times 7$ pounds' weight $= 84$ pounds' weight, and the tension of the string $= 20 \times 7$ pounds' weight $= 140$ pounds' weight.

(*b*) *Second Method.*—The values of X and Y may be determined by Lamé's Theorem, the angles between the different pairs of forces being known.

Numerical Example.—Taking the same example as before, let P pounds' weight be the force applied, and T pounds' weight the tension of the string, and let α be the angle ABN.

Then

$$\frac{P}{\sin(90^\circ + \alpha)} = \frac{T}{\sin 90^\circ} = \frac{112}{\sin(180^\circ - \alpha)}$$

i.e.

$$\frac{P}{\cos \alpha} = \frac{T}{1} = \frac{112}{\sin \alpha}$$

$$\therefore \begin{cases} P = 112 \cot \alpha = 112\left(\frac{3}{4}\right) = 84 \\ T = 112 \operatorname{cosec} \alpha = 112\left(\frac{5}{4}\right) = 140 \end{cases}$$

(*c*) *Third Method.*—We may resolve in two different directions, and thus obtain two simple equations to find X and Y.

If we require X and not Y, we may resolve in a direction at right angles to the force Y, which does not then enter into the resulting

equation. Thus, if the dotted line in Fig. 36 is at right angles to the direction of the force Y, we have

$$X \cos \gamma = P \cos \beta$$

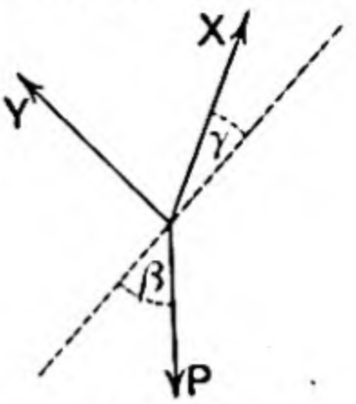

FIG. 36.

Numerical Example.—Taking again the same numerical example, and resolving vertically, we have

$$T \sin \alpha = 112$$
$$\therefore\ T = 112 \operatorname{cosec} \alpha = 140$$

Also, resolving horizontally,

$$P = T \cos \alpha = 140 \left(\frac{3}{5}\right) = 84$$

EXAMPLES V

1. Three forces of given magnitudes keep a particle in equilibrium. Determine the angle between the last two forces when the measures of the forces are :—

(i) 6, 11, 18	(v) 17, 15, 8
(ii) 5, 3, 4	(vi) 13, 15, 7
(iii) 7, 8, 5	(vii) 13, 15, 8
(iv) 22, 29, 51	(viii) 13, 8, 7

2. A mass of weight W is supported by two fine light strings inclined at angles α and β with the vertical. Determine the tensions of the strings.

3. A body, which weighs 12 pounds, is kept at rest by means of two fine light cords, one being horizontal, and the other inclined to the horizontal at an angle whose tangent is $\frac{3}{4}$; find their tensions.

4. A mass of 20 pounds is supported by two fine light strings inclined at angles of 30° and 60° with the vertical. Determine the tensions of the strings.

5. A mass of 5 pounds is supported by two fine light strings, each inclined at an angle of 75° with the vertical. Determine the tensions of the strings.

6. A mass of 100 pounds is supported by two fine light strings inclined at angles of 45° and 15° with the vertical. Determine the tensions of the strings.

7. A force of given magnitude P is in equilibrium with two forces each of given magnitude Q. Show that the two equal forces must be inclined at an angle $2 \cos^{-1} \frac{P}{2Q}$.

8. A force of 40 pounds' weight is in equilibrium with two equal forces, the cosine of the angle between the given force and one of the equal forces being $-\frac{3}{5}$. Determine the magnitude of each of the equal forces.

9. A fine light string AOB, with its extremities fixed at points A and B, is thrown into a state of tension by a force applied at O. If the line of action of the applied force passes through the middle point of AB, show that the tensions of the two portions of the string are proportional to their lengths.

10. A fine light string AOB, of length 7 inches, attached at its

extremities to two points A and B, situated 5 inches apart, is thrown into a state of tension by a force of 10 pounds' weight applied at a point O of the string distant 3 inches from A. If the line of action of the applied force passes through the middle point of AB, find the tensions of the strings AO and OB.

11. In question 9, if AOB is a right angle, and if the force is applied in a direction perpendicular to AB, show that the tensions of the two portions of the string are inversely proportional to their lengths.

12. In question 10, if the force of 10 pounds' weight is applied in a direction perpendicular to AB, find the tensions of the strings AO and OB.

13. A fine light string AOB, of length $7\frac{3}{4}$ feet, is attached at its extremities to two points A and B, situated $6\frac{1}{4}$ feet apart in a horizontal line. From a point O of the string, distant 6 feet from A, a mass of 100 pounds is suspended. Find the tensions of the strings AO and OB.

14. A fine light string AOB, of length l, is attached at its extremities to two points A and B, situated at a distance a apart in a horizontal line. Find the tension of the string when a mass of weight W is suspended from its middle point.

15. A heavy particle is suspended by two equal strings attached to points in the same horizontal line. If the lengths of the strings are equally increased, show that their tensions are diminished.

16. In the preceding question, if the lengths of the strings are altered so that they are afterwards inclined to the vertical at the same angle at which they were formerly inclined to the horizontal, show that the tension of each is altered inversely as its length.

17. A heavy particle is suspended from the middle point of a string 29 inches in length, attached at its extremities to two points situated 21 inches apart in the same horizontal line. If the length of the string is increased to 35 inches, show that its tension is diminished in the ratio 116 : 100.

18. A and B are two fixed points situated in a horizontal line at a distance c apart. Two fine light strings AC and BC, of lengths b and a respectively, support a mass at C. Show that the tensions of the strings are in the ratio $b(a^2+c^2-b^2) : a(b^2+c^2-a^2)$.

19. A and B are two fixed points distant 4 feet apart in a horizontal line. Two fine light strings AC and BC, of lengths 2 and 3 feet respectively, support at C a mass of weight W. Find the tensions of the strings.

20. A and B are two fixed points situated in a horizontal line at a distance c apart. A fine light string ACB, of length $2c$, supports at C a mass of weight W. If the difference between the tensions of the two parts of the string is $\frac{1}{4}W$, show that the difference between their lengths is $\frac{1}{7}c$.

21. A fine light string, of length 64 inches, has its extremities attached to two fixed points, situated 40 inches apart in a horizontal line. If the string can bear any tension up to 20 pounds' weight, find the greatest load that can be supported at a point of the string distant 25 inches from one extremity.

22. A fine light string OAB is attached at one extremity to a fixed point O, and supports at the other extremity B a mass of given weight W. To a point A of the string, situated at a given distance a from O, a force of given magnitude P is applied in a given direction making an angle α with the upward vertical. Show that, in the position of equilibrium, OA is inclined to the vertical at an angle $\tan^{-1}\frac{P\sin\alpha}{W-P\cos\alpha}$, and determine the tension of OA.

23. A fine light string OAB is attached at one extremity to a fixed point O, and supports at the other extremity B a mass of 32 pounds. To a point A of the string, distant 14 inches from O, a force of 12 pounds' weight is applied in a direction making an angle of 60° with the upward vertical. Find the vertical distance of A below O in the position of equilibrium, and determine the tension of OA.

24. A fine light string OAB is attached at one extremity to a fixed point O, and supports at the other extremity B a mass of given weight W. It is pulled aside from the vertical by means of a force applied to a given point A of the string. Show how to find the magnitude and direction of the smallest force which will cause the string to rest in a given position.

25. A fine light string OAB is attached at one extremity to a fixed point O, and supports at the other extremity B a mass of 10 pounds. It is pulled aside, until OA makes an angle of 30° with the vertical, by means of a force applied at A. Find the magnitude and direction of the smallest force which will keep the system in this position.

26. In the preceding question, if the applied force is altered so as to be inclined at an angle of 30° to the vertical, the position of equilibrium being unaltered, show that the tension of the string is diminished in the ratio 3 : 2.

27. In the preceding question, if the applied force is again altered, so as to be horizontal, show that the tension of the string is doubled.

28. A particle, of mass 7 pounds, is suspended from a fixed point by a fine light string 21 inches in length, and is drawn aside until it rests at a vertical distance of 9 inches below the horizontal through the fixed point. It is supported in this position by a force of 7 pounds' weight applied to the particle. Find the tension of the string.

29. In the preceding question, if the mass is afterwards supported in the same position by a force of 11 pounds' weight, show that the tension of the string is doubled.

30. Three forces P, Q, R, acting at a point, are in equilibrium, and the angle between P and Q is double of the angle between P and R. Prove that $R^2=Q(Q-P)$.

31. If three forces P, Q, R, acting at a point, are in equilibrium, and if $R^2=Q(Q-P)$, prove that the angle between P and Q is double of the angle between P and R.

32. $ABCD$ is a quadrilateral, having the angles at A and D right angles, and $CB=CD$. Forces P, Q, R, acting along AB, CA, AD respectively, are in equilibrium. If $P=\frac{2}{\sqrt{3}}R$, show that $AB=\frac{1}{2}BC$ or $\frac{3}{2}BC$.

33. A given force P is in equilibrium with two others, one of which is of given magnitude Q, while the other is of unknown magnitude X in a given direction inclined at an angle α with the force P, all three forces acting at the same point. Show how to determine the direction of the force Q and the magnitude of the remaining force.

Show that there are in general two solutions, and that if $P > Q$, there are limits to the angle α beyond which the question is impossible of solution.

34. In the preceding question, show that the sum of the two values of X is $-2P\cos\alpha$, and their product is $P^2 - Q^2$.

35. A fine light string OAB is attached at one extremity to a fixed point O, and supports at its other extremity B a mass of 15 pounds. It is pulled aside, until OA makes an angle of 60° with the vertical, by means of a force of 13 pounds' weight applied at A. Find the tension of the string.

36. If three forces, acting at a point, are in equilibrium, and if straight lines are drawn perpendicular to their lines of action so as to form a triangle, then the sides of the triangle are proportional to the forces to which they are respectively perpendicular. Cf. Art. 41.

37. If three forces P, Q, R, acting at a point O, are in equilibrium, and if a circle through O cuts their lines of action in A, B, C respectively, then $P : Q : R = BC : CA : AB$.

48. Ex. 6.—*A given force P, acting along a given straight line OD, is in equilibrium with two other forces of unknown magnitudes X and Y. The force X acts along a given straight line OE, and the force Y through a given point F. It is required to determine the values of X and Y, and the direction of the force Y.*

(i) *By Geometrical Construction.*—The lines of action of the three forces must be concurrent. Hence, joining FO, the line so drawn is the line of action of the force Y.

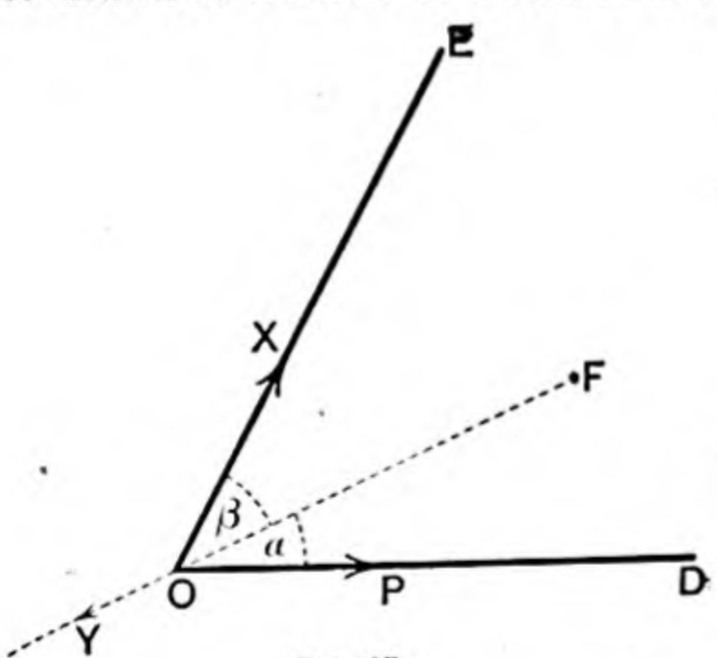

FIG. 37.

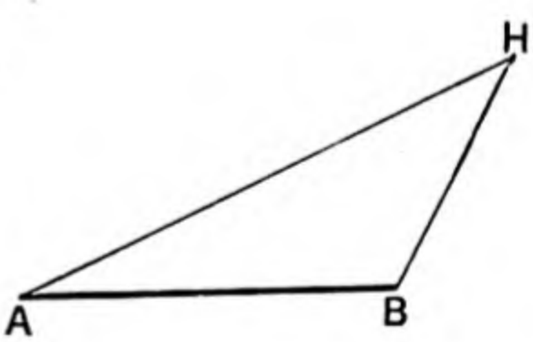

FIG. 37a.

The question now becomes the same as that of Art. 47, and is solved in the same way.

(ii) *By Calculation*—(*a*) *First Method.*—The position of the point F being known relatively to the lines OD and OE, the angles FOD and FOE may be determined with the aid of geometry and trigonometry. Let these angles be α and β respectively.

Then, by Lamé's theorem,

$$\frac{X}{\sin(\pi-\alpha)}=\frac{Y}{\sin(\alpha+\beta)}=\frac{P}{\sin(\pi-\beta)}$$

$$\therefore \begin{cases} X=P\sin\alpha\ \operatorname{cosec}\beta, \\ \text{and } Y=P\sin(\alpha+\beta)\operatorname{cosec}\beta. \end{cases}$$

Numerical Example.—Fig. 38 represents a straight rod AFB, of no appreciable weight, capable of turning freely in a vertical plane about the fixed point F, the distances AF and FB being 6 and 5 feet respectively. A mass of 30 pounds is suspended from B, and the rod is supported in a horizontal position by a fine light string AC, inclined at an angle of 30° to the horizontal. Suppose that it is required to determine the tension of the string and the force of constraint at F.

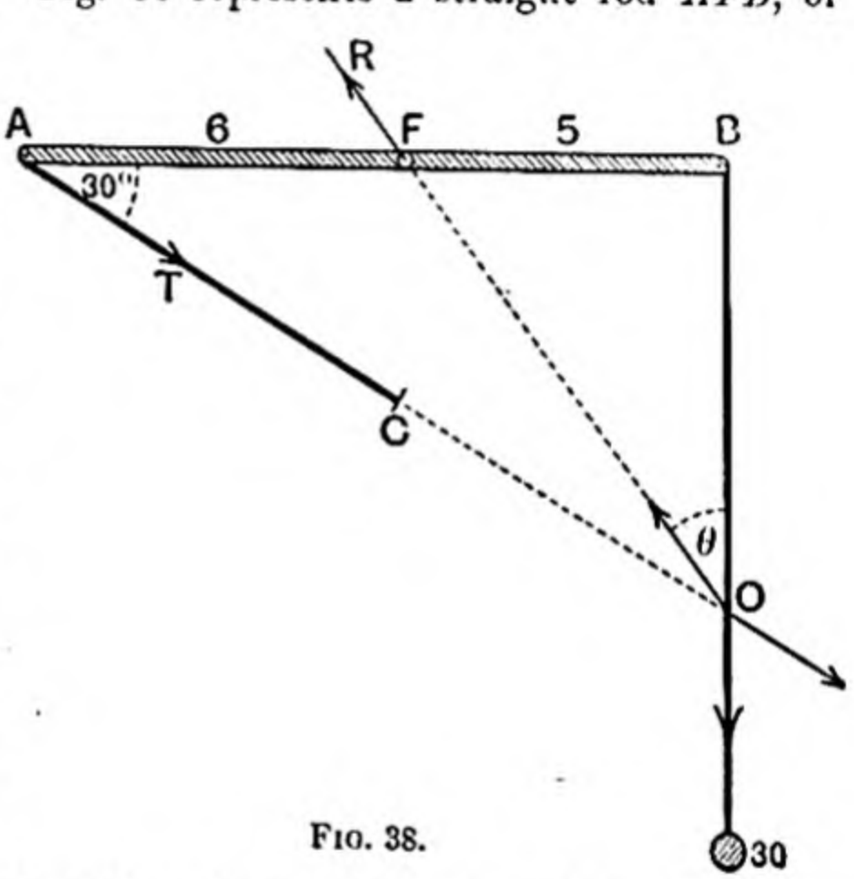

FIG. 38.

Let AC meet the vertical through B in O. Then OF is the line of action of the force of constraint at F. Let $\widehat{FOB}=\theta$.

Then $$\tan\theta=\frac{FB}{BO}=\frac{FB}{AB\tan 30^\circ}=\frac{5}{11\tan 30^\circ}=\frac{5\sqrt{3}}{11},$$

whence $$\sin\theta=\frac{5\sqrt{3}}{14} \text{ and } \cos\theta=\frac{11}{14}.$$

Let T pounds' weight be the tension of the string, and R pounds' weight the force of constraint at F.

Then, by Lamé's theorem,

$$\frac{T}{\sin\theta}=\frac{R}{\sin 60^\circ}=\frac{30}{\sin(60^\circ-\theta)}$$

$$\therefore T=\frac{30\sin\theta}{\sin(60^\circ-\theta)}=\frac{30}{\sin 60^\circ\cot\theta-\cos 60^\circ}=\frac{30}{\frac{\sqrt{3}}{2}\cdot\frac{11}{5\sqrt{3}}-\frac{1}{2}}$$

$$=50.$$

$$\text{Also } R = \frac{30 \sin 60^\circ}{\sin (60^\circ - \theta)} = \frac{30}{\cos\theta - \cot 60^\circ \sin\theta} = \frac{30}{\frac{11}{14} - \frac{1}{\sqrt{3}} \cdot \frac{5\sqrt{13}}{14}} = 70.$$

Thus the tension of the string is 50 pounds' weight, and the force of constraint at F is 70 pounds' weight inclined at an angle $\cos^{-1} \frac{11}{14}$ with the vertical.

(*b*) *Second Method.* — The position of the point F being known relatively to the lines OD and OE, the lengths of the perpendiculars from F upon the lines OD and OE may be determined with the aid of geometry and trigonometry. Let p and q be the measures of these perpendiculars respectively. Then we have $Xq = Pp$. This equation determines X.

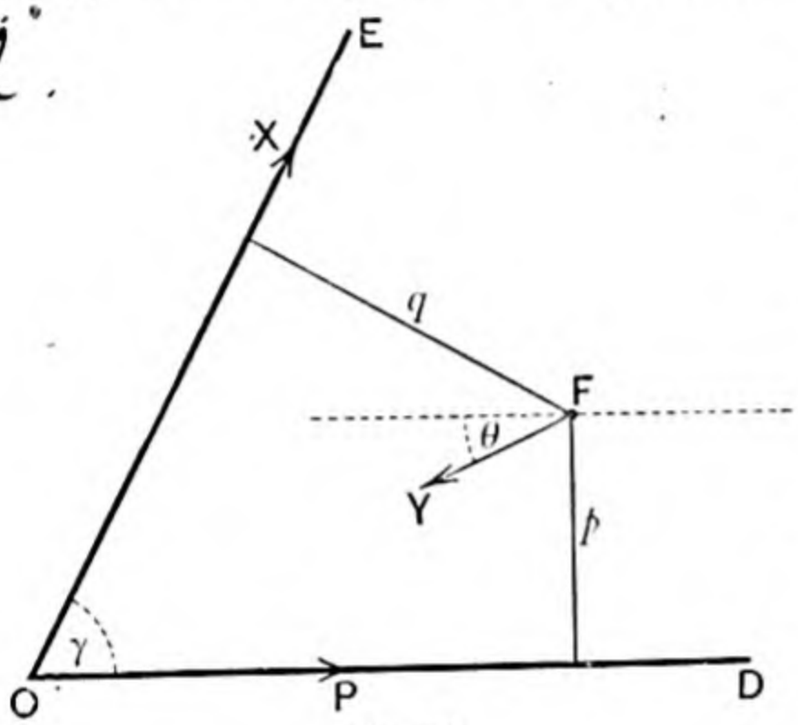

FIG. 39.

Let γ be the known angle between OD and OE, and suppose that the direction of Y makes an angle θ with the direction DO, as indicated in the figure. Then, resolving along and perpendicular to OD, we have

$$\begin{cases} Y\cos\theta = P + X\cos\gamma \\ Y\sin\theta = X\sin\gamma. \end{cases}$$

As X has already been determined, these two equations are sufficient to determine Y and θ.

Numerical Example. — Taking the same numerical example as before, the length of the perpendicular from F upon AC is

$6 \sin 30^\circ$ feet
$= 3$ feet.

Hence we have

$T \times 3 = 30 \times 5,$
$\therefore\ T = 50.$

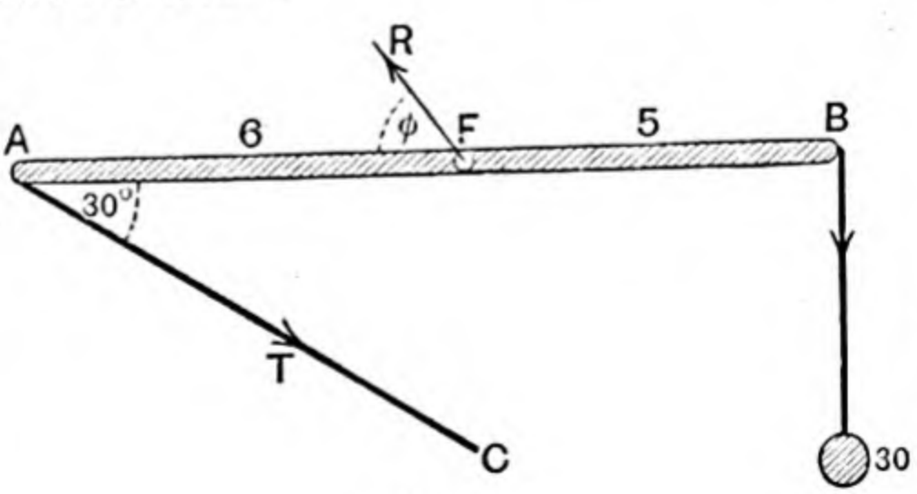

FIG. 40.

Let ϕ be the angle that the constraint at F makes with BA. Then, resolving vertically and horizontally, we have

$$\begin{cases} R\sin\phi = 30 + T\cos 60^\circ = 30 + 25 = 55. \\ R\cos\phi = T\cos 30^\circ = 25\sqrt{3}. \end{cases}$$

$$\therefore\ R^2 = 55^2 + (25\sqrt{3})^2 = 5^2\{121 + 75\} = 5^2 \times 14^2.$$

$$\therefore\ R = 70.$$

Also
$$\sin\phi = \frac{55}{R} = \frac{55}{70} = \frac{11}{14}.$$

Thus we have the same result as before.

49. Centre of Gravity.—In the following examples, we assume that the resultant weight of a rigid body acts along a vertical line, which, in all positions of the body, passes through a point fixed relatively to the body. This point may thus be regarded as the point of application of the resultant weight of the body, and is called its *Centre of Gravity*. We shall consider this subject in Chapter VIII.

We also assume that if a *uniform* body possesses a centre of figure, then its centre of gravity coincides with its centre of figure.

50. Smooth Hinge.—In some of the following examples, a body is supposed to have one point A constrained to remain in a fixed position. The constraint is supposed to be such as to compel the point A of the body to remain in its fixed position by means of a single force applied at A. This force is self-adjusting as regards magnitude and direction, and accommodates itself so as, if possible, to preserve equilibrium. Under these circumstances the body is described as *smoothly hinged* to the fixed point A, or as *capable of turning freely* about the fixed point A.

51. Ex. 7.—*A straight rod AB, of weight W, is capable of turning freely about one extremity A, and is pulled aside from the vertical by a given horizontal force P applied at B. Given the position of G, the centre of gravity of the rod, determine the angle at which the rod is inclined to the vertical in the position of rest, and find the magnitude and direction of the action at A.*

Let θ be the angle at which the rod is inclined to the vertical, and let the horizontal through B meet the vertical through A in N.

The weight of the rod is equivalent to a single force W, acting vertically downwards through G. Let AG and BG be of lengths a and b respectively.

[*N.B.*—If the rod is uniform, G is the middle point of AB.]

The forces acting upon the rod have been reduced to three; namely, the force P acting horizontally through B, the force W acting vertically through G, and the reaction R of the hinge at A acting at an angle ϕ with the horizontal.

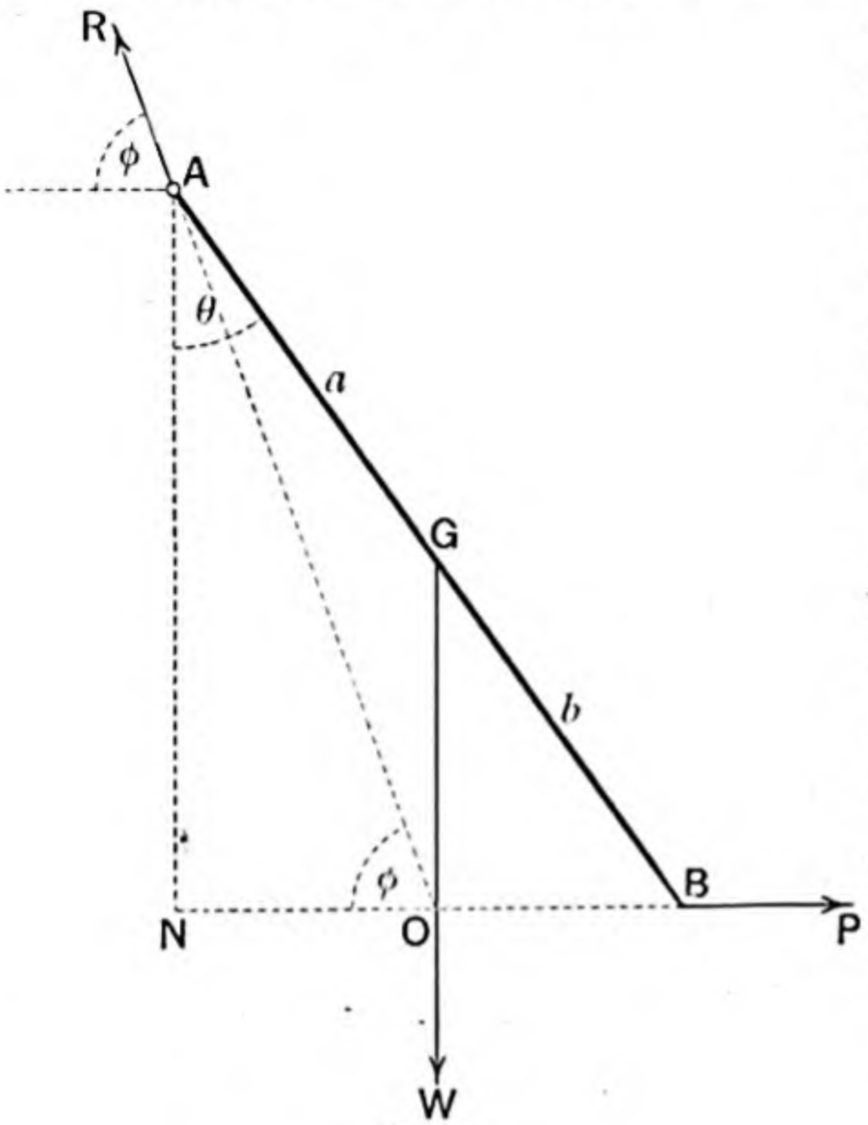

FIG. 41.

First Method.—Let NB meet the vertical through G in O. Then OA is the line of action of the force R, and $\widehat{NOA} = \phi$. By Lamé's Theorem we have

$$\frac{P}{\cos\phi} = \frac{W}{\sin\phi} = \frac{R}{1}$$

$$\therefore \quad \begin{cases} \tan\phi = \dfrac{W}{P} \\ \text{and } R = W \operatorname{cosec}\phi = W\sqrt{1+\cot^2\phi} = \sqrt{W^2+P^2}. \end{cases}$$

Now $$NO : NB = a : a+b$$

i.e. $$AN\cot\phi : AN\tan\theta = a : a+b$$

$$\therefore \quad \tan\theta = \frac{a+b}{a}\cot\phi$$

$$= \frac{a+b}{a}\cdot\frac{P}{W}.$$

Thus the rod rests at an angle $\tan^{-1}\left(\dfrac{a+b}{a}\cdot\dfrac{P}{W}\right)$ with the vertical, and the action at A is $\sqrt{W^2+P^2}$ in a direction making an angle $\tan^{-1}\dfrac{W}{P}$ with the horizontal.

Second Method (without joining AO).—The perpendiculars from A upon the lines of action of P and W are of lengths $(a+b)\cos\theta$ and $a\sin\theta$ respectively.

Hence $$P(a+b)\cos\theta = Wa\sin\theta$$

$$\therefore \tan\theta = \frac{a+b}{a}\cdot\frac{P}{W}.$$

Again, resolving vertically and horizontally, we have

$$\begin{cases} R\sin\phi = W \\ R\cos\phi = P \end{cases}$$

$$\therefore R = \sqrt{W^2+P^2}, \text{ and } \tan\phi = \frac{W}{P}.$$

Thus we obtain the same result as before.

*52. Ex. 8.—*An endless fine light string, in the shape of a triangle, is thrown into a state of tension by means of three forces applied at its angular points. It is required to determine the tensions of the different parts of the string.*

(i) *By Geometrical Construction.*—Let the sides of the triangle be denoted by a, b, c, and its angular points by bc, ca, ab.

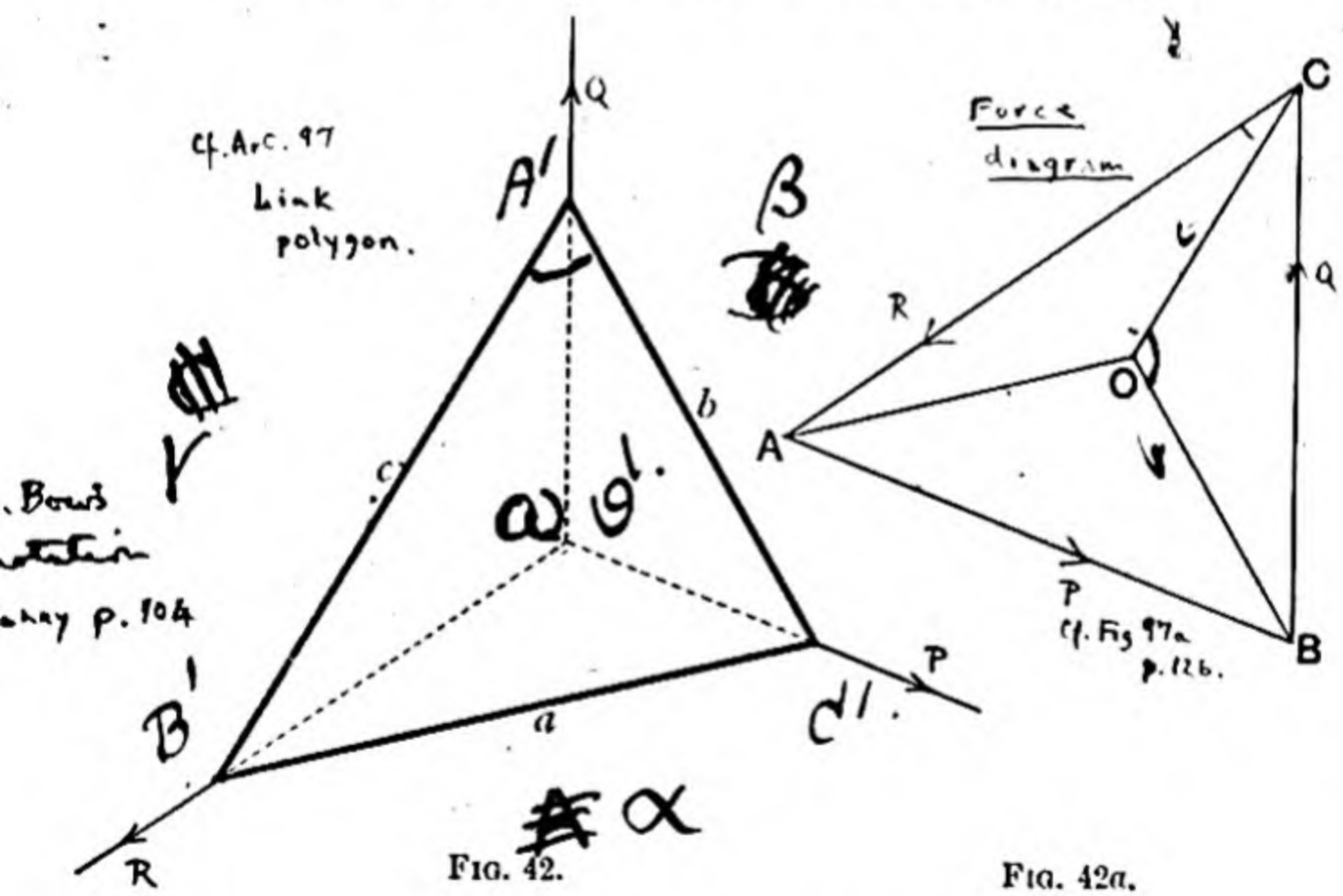

FIG. 42. FIG. 42a.

The applied forces will act along lines which meet at a point within the triangle, and will be represented by the sides of another triangle ABC taken one way round, BC, CA, AB representing the forces applied at the points bc, ca, ab respectively. We will suppose that these forces are all known, and that the triangle ABC has been constructed.

Draw BO and CO parallel to the strings b and c respectively, to meet in O, and join OA.

Then *BCOB* (this way round) is the triangle of forces for the point *bc*, so that *CO* represents the pull of the string *c* at the end *bc*. Hence *OC* represents the pull of the string *c* at the end *ca*. Thus *OCAO* (this way round) must be the triangle of forces for the point *ca*.

Hence the straight lines drawn through *A*, *B*, *C* parallel to the strings *a*, *b*, *c* respectively meet at a point *O*, and the tensions of the strings *a*, *b*, *c* are given by *OA*, *OB*, *OC* respectively.

(ii) *By Calculation.*—The lengths of the lines *OA*, *OB*, *OC* may be determined by trigonometry, and thus the tensions of the strings become known ; or, without drawing a force diagram, we may determine the tensions of the strings by applying Lamé's Theorem successively to the consideration of the equilibrium of each angular point of the triangle.

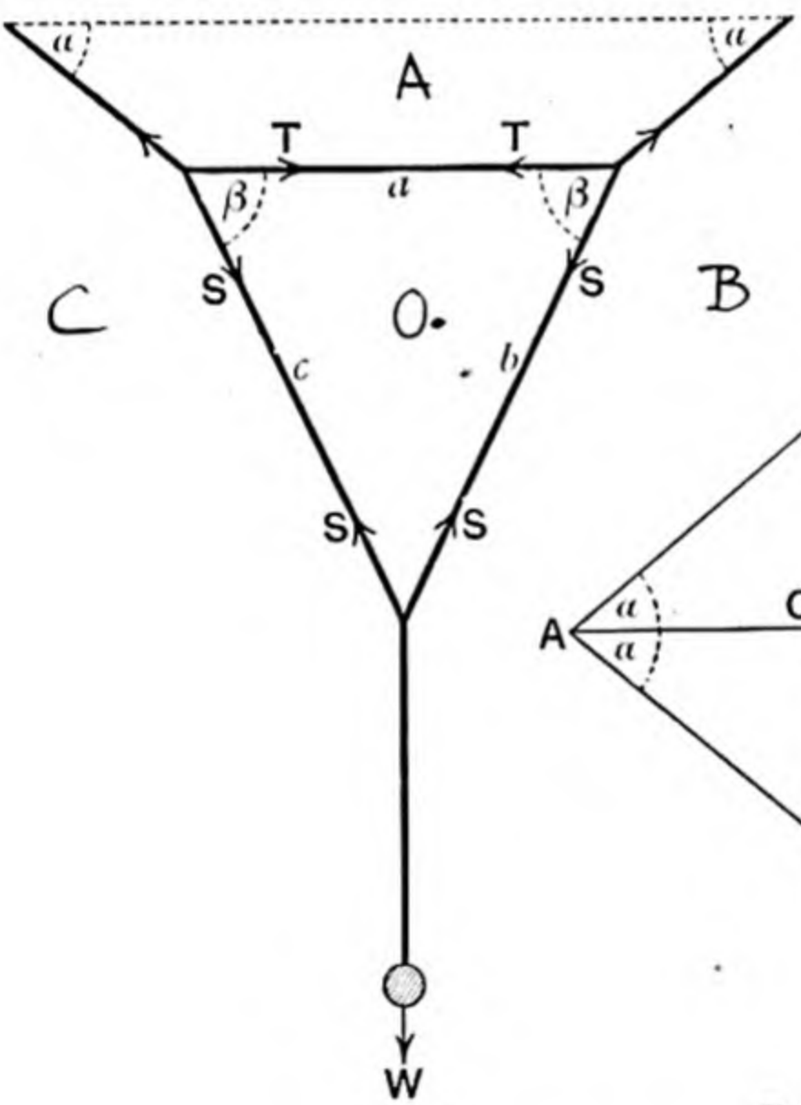

FIG. 43.

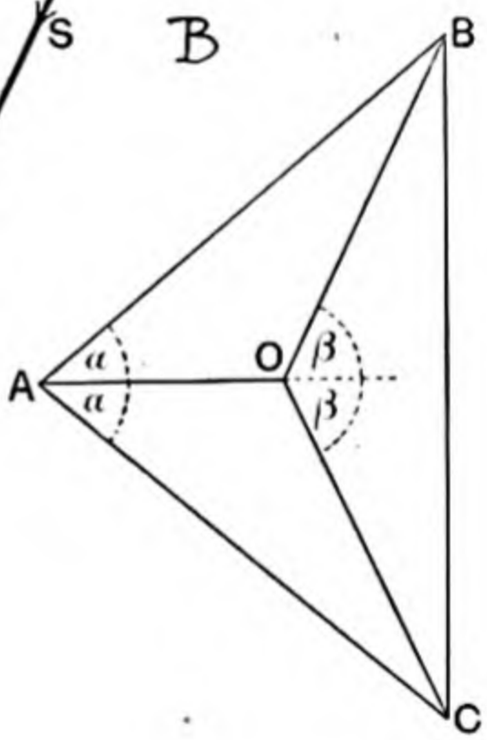

FIG. 43*a*.

Particular Example.—Fig. 43 represents a symmetrical arrangement of fine light strings, supporting a mass of weight *W*, the dotted line being horizontal, and the notation being the same as that described above. Suppose that it is required to determine the tension of the string *a*.

In the force diagram, *BC* is of length *W* and the perpendiculars

from A and O upon BC are of lengths $\frac{1}{2}W \cot \alpha$ and $\frac{1}{2}W \cot \beta$ respectively.

Hence AO is of length $\frac{1}{2}W(\cot \alpha - \cot \beta)$. Thus the tension of the string a is $\frac{1}{2}W(\cot \alpha - \cot \beta)$.

Otherwise.—Let S be the tension of each of the strings b and c, and T the tension of the string a.

Then resolving vertically upwards for the equilibrium of the knot bc, we have

$$2S \sin \beta = W$$
$$\therefore S \sin \beta = \tfrac{1}{2}W.$$

Also, using Lamé's Theorem for the equilibrium of the knot ca, we have

$$\frac{T}{\sin(\pi - \beta + \alpha)} = \frac{S}{\sin(\pi - \alpha)}$$
$$\therefore T = S \sin(\beta - \alpha) \operatorname{cosec} \alpha$$
$$= S \sin \beta \cdot \frac{\sin(\beta - \alpha)}{\sin \alpha \cdot \sin \beta}$$
$$= S \sin \beta (\cot \alpha - \cot \beta)$$
$$= \tfrac{1}{2}W(\cot \alpha - \cot \beta).$$

*EXAMPLES VI

1. A uniform thin rod AB, which can turn freely in a vertical plane about a hinge at A, is kept in a horizontal position by a string BC attached to a fixed point C in the vertical plane, the angle ABC being obtuse. Show in a diagram the forces acting on the rod, and prove that two of them are equal.

2. A fine uniform rod, 7 feet 6 inches long and weighing 20 pounds, can turn freely about its lower end O. A fine light *horizontal* string AB connects a point A of the rod, distant 6 feet 8 inches from O, with a fixed point B, situated 4 feet vertically above O. Find the tension of the string and the reaction at O.

3. A uniform beam, 12 feet in length, has a fixed hinge at one end, and is supported by a fine light cord, 13 feet long, attached to the other end and to a fixed point situated 20 feet vertically above the hinge. Find the tension of the cord, assuming that the beam weighs 140 pounds.

4. A drawbridge, consisting of a uniform plank, of length 10 feet and weighing 240 pounds, which is movable in a vertical plane about a smooth hinge at one of its ends, is kept in a nearly vertical position by means of a rope which is fastened to the other end and is also made fast to a peg, situated vertically over the hinge and at a height of 24 feet above it. Find the limit to which the tension in this rope diminishes as the bridge approaches the vertical position.

5. A uniform rod AB, of weight W, is capable of turning freely about the extremity A, which is fixed, and rests in a position inclined at an angle of 60° to the vertical, being supported by a force applied at the lower end B. Find the magnitudes of the applied force and of the reaction at A, (i) when the applied force is horizontal, and (ii) when it is inclined at an angle of 30° to the horizontal.

6. In the preceding example, find the magnitude and direction of

the smallest force which, applied at B, will keep the rod at rest in the given position; and determine in this case the value of the reaction at A.

7. A thin uniform rod AB, of length 10 feet and weighing 12 pounds, is capable of turning freely about a smooth hinge at A. It is supported by a fine string, of length 10 feet, connecting B with a point C, situated 16 feet from A in a horizontal line. Find the tension of the string and the action at the hinge.

8. A straight rod AB, of length $4\frac{1}{2}$ feet and weighing 24 pounds is capable of turning freely about one extremity A, and is pulled aside from the vertical by a horizontal force of 10 pounds' weight applied at B. If the centre of gravity of the rod is at a point 2 feet from B, find the distance of B from the vertical through A in the position of equilibrium, and determine the magnitude of the action at A.

9. In the preceding example, find the smallest force which, applied at B instead of the horizontal force of 10 pounds' weight, will keep the rod at rest in the same position.

10. A fine light rod AD, of length 2 feet and of no appreciable weight, is capable of turning freely in a vertical plane about its lower extremity A. A fine light string, also of length 2 feet, connects a point C of the rod, distant 1 foot 6 inches from A, with a point B situated 2 feet 6 inches vertically above A. Find the tension of the string when a mass of weight W is suspended from D.

11. A fine light rod AD, of length 2 feet and of no appreciable weight, is capable of turning freely in a vertical plane about its lower extremity A. A fine light string, of length 1 foot 6 inches, connects a point C of the rod, distant 1 foot 3 inches from A, with a point B situated 1 foot 8 inches vertically above A. What mass must be suspended from D in order that the tension of the string may be 144 pounds' weight.

12. A straight rod AB, of no appreciable weight, is capable of turning freely in a vertical plane about a fixed point F which divides AB in the ratio 2 : 1. A mass of weight W is suspended from B, and the rod is supported in a horizontal position by a fine string AO, connecting A with a point O situated vertically below F. If the string makes an angle of 30° with the horizontal, determine its tension and the action at F.

13. A straight rod AFB, of length 6 feet 3 inches, and of no appreciable weight, is capable of turning freely in a vertical plane about a fixed point F, situated 2 feet 6 inches from B. A mass of 12 pounds is suspended from B, and the rod is supported in a horizontal position by a fine string AC, connecting A with a point C situated 2 feet vertically below F. Find the tension of the string and the action at F.

14. A uniform lamina in the shape of an equilateral triangle, of weight W, can turn freely in a vertical plane about a fixed hinge at A. It is kept with AB horizontal, and C downwards, by a fine string connecting C with a point D in the same horizontal line as A, and at a distance from A equal to twice the side of the triangle. Find the tension of the string CD, and the reaction of the hinge at A.

15. A uniform square lamina $ABCD$, of weight W, is capable of

turning freely in a vertical plane about a hinge at A. Find what horizontal force, applied at C, will keep the lamina in equilibrium with BC horizontal and below AD. Determine also the action of the hinge at A.

16. A uniform square lamina $ABCD$, of weight W, is capable of turning freely in a vertical plane about a hinge at A. Find what horizontal force, applied at D, will keep the lamina in equilibrium with AC inclined to the downward vertical at an angle of 15°.

17. A uniform square lamina $ABCD$, of weight W, is capable of turning freely about the point A, which is fixed; a fine string, of length equal to the side of the square, connects B with a point E, situated in a horizontal line with A. If $\widehat{BAE} = \alpha$, show that the tension of the string is

$$\tfrac{1}{4}W(\operatorname{cosec}\alpha - \sec\alpha).$$

18. $ABCDEF$ is a regular hexagon. A given force P, acting along AE, is in equilibrium with two forces of unknown magnitudes, one of which acts along FB, while the other acts at C in an unknown direction. Determine the magnitudes of the unknown forces.

19. A uniform straight rod AB, of weight W, is suspended from a point C by two fine light strings AC and BC at right angles. If $\widehat{CAB} = \alpha$, prove that in the position of equilibrium the rod rests inclined at an angle 2α to the vertical, and that the tensions of the strings AC and BC are $W\cos\alpha$ and $W\sin\alpha$ respectively.

20. A uniform rod, of length 7 feet and weighing $5\frac{1}{2}$ pounds, is suspended from a fixed point by two fine light strings, of lengths 6 and 7 feet, connecting the fixed point with the extremities of the rod. Find the tensions of the strings and the cosine of the angle at which the rod is inclined to the vertical in the position of equilibrium.

21. A straight rod AB, whose centre of gravity is 7 inches from A and 5 inches from B, is suspended from a fixed point C by means of two strings AC and BC, of lengths 13 and 7 inches respectively. Find the angle that the rod makes with the vertical in the position of equilibrium, and determine the tensions of the strings in terms of W, the weight of the rod.

22. A fine rod OA, one foot in length and of no appreciable weight, is capable of turning freely about the extremity O, which is fixed. A mass of weight W is attached at A, and the rod is supported in a horizontal position by means of a fine light rod BC, of length 10 inches, smoothly hinged at B to the rod OA and at C to a point situated 6 inches vertically below O. Find the action at O and the thrust in BC.

23. Find a point P within a triangle ABC, so that forces represented by $\lambda\,(PA)$, $\mu\,(PB)$, $\nu\,(PC)$ may be in equilibrium.

Make use of this to prove the geometrical theorem, that if points D, E, F are taken in the sides BC, CA, AB respectively of a triangle ABC, such that

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1,$$

then AD, BE and CF are concurrent.

24. A straight uniform rod AB, of weight W and of length c, is capable of turning freely in a vertical plane about one extremity A. A fine light string of length a connects the extremity B with a point C, fixed at a distance b vertically above A. Show that the tension of the string is $\frac{a}{2b}W$, and the reaction at A is

$$\frac{\sqrt{2b^2+2c^2-a^2}}{2b}W.$$

25. A straight rod AD, of no appreciable weight and of length d, is capable of turning freely in a vertical plane about one extremity A. The point B of the rod, at a distance c from A, is connected by a light inextensible string of length a to a point C, fixed at a distance b vertically above A. At D is attached another fine light string supporting a mass of weight W. Show that the tension of the string is $\frac{ad}{bc}W$, and the reaction at A is

$$\frac{\sqrt{(a^2d-b^2c)(d-c)+c^3d}}{bc}W.$$

26. A straight rod BAD, of no appreciable weight, is capable of turning freely in a vertical plane about a fixed point A, where AB and AD are of lengths c and d respectively. A mass of weight W is suspended from D, and the rod is supported by a fine string BC, of length a, connecting the extremity B with a point C, fixed at a distance b vertically below A. Show that the tension of the string is $\frac{ad}{bc}W$, and the reaction at A is

$$\frac{\sqrt{(a^2d+b^2c)(c+d)-c^3d}}{bc}W.$$

27. A uniform rod is capable of turning freely about one extremity, which is fixed, and rests in a position inclined to the vertical, being supported by a horizontal force applied at the lower end. Show that, if the direction of the applied force be altered, and its magnitude adjusted so as not to disturb the position of equilibrium, then when the force is diminished as much as possible, the reaction of the hinge is diminished in the same ratio, and its direction altered so that it now makes the same angle with the vertical that it formerly made with the rod.

28. Four fine light strings are knotted together to form a rhombus $ABCD$, which is suspended from A. A mass of weight W is attached at C, while B and D are kept apart in a horizontal line by two equal and opposite forces acting at B and D. If each side of the rhombus is inclined to the vertical at an angle α, determine the value of each of the applied forces, and find the tension of the string.

29. An endless fine light string, of length 16 inches, is in the shape of a triangle ABC, the sides AC and BC being each of length 5 inches. A mass of 24 pounds is suspended from C, and the whole rests with AB horizontal, being supported by forces applied at A and B in directions perpendicular to BC and CA respectively. Find the

magnitude of each of the applied forces, and determine the tensions of the different parts of the string.

30. An endless fine light string is in the shape of a triangle ABC, the angles at A and B being each of magnitude 2α. A mass of weight W is suspended from C, and the whole is supported, with AB horizontal, by forces applied at A and B along the bisectors of the angles A and B respectively. Show that the tension of the string is the same throughout, and determine its magnitude and that of each of the applied forces.

31. An endless fine light string in the shape of a triangle ABC is thrown into a state of tension by means of three forces applied at the angular points along lines which intersect at a point O within the triangle. The circles circumscribed to ABC and OBC meet AO produced in O' and A' respectively. Show that the forces applied at A, B, C and the tensions of BC, CA, AB are proportional to BC, CA', $A'B$, $O'A'$, $O'B$, $O'C$ respectively.

32. In the preceding example, if O is the orthocentre of the triangle ABC, prove that the forces applied at A, B, C and the tensions of BC, CA, AB are proportional to BC, CA, AB, OA, OB, OC respectively.

33. An endless fine light string is in the shape of a triangle ABC. A mass of weight W is suspended from C, and the whole rests with AB horizontal, being supported by forces applied at A and B in directions perpendicular to BC and CA respectively. Show that the tension of AB is $W \cot C$.

34. An endless fine light string is in the shape of a quadrilateral $ABCD$, AD and BC being parallel and AD longer than BC. Masses each of weight W are suspended from B and C, and the whole rests with AD horizontal, being supported by forces applied at A and D, each at an angle α with the horizontal. If AB and CD are each inclined to the horizontal at an angle β, show that the tension of AD is

$$W(\cot\alpha - \cot\beta).$$

CHAPTER IV

CONCURRENT FORCES

53. Resultant of Concurrent Forces.—*To find the resultant, by geometrical construction, of any given system of concurrent forces.*

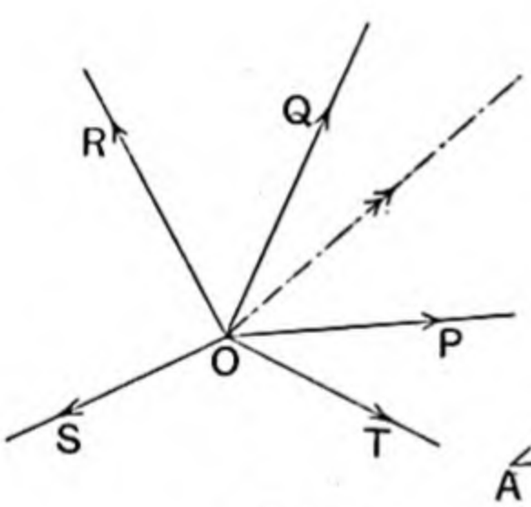

FIG. 44.

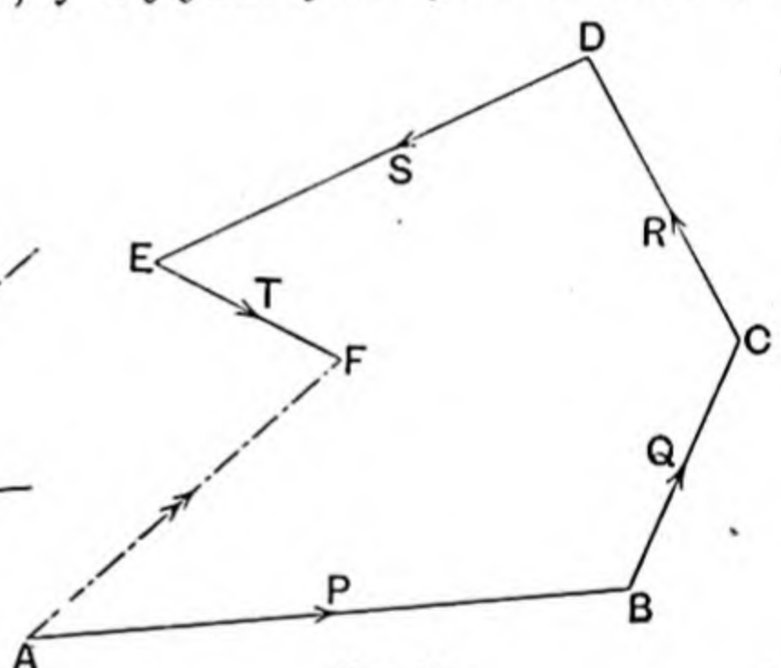

FIG. 44a.

Let a number of given forces P, Q, R, S, T act in one plane along known lines which intersect at the point O.

From any suitable point A, and with any suitable scale, draw AB to represent the force P. From B draw BC to represent the force Q. Similarly, draw CD, DE, EF to represent the forces R, S, T respectively.

We shall prove that the straight line from A, where we started, to F, where we finished, represents the resultant in magnitude and direction, and that its line of action passes through O.

The two forces P and Q are represented by AB and BC; hence, by the extension of the Parallelogram of Forces, their resultant is represented by AC and acts through O.

Replacing the forces P and Q by their resultant, we see that the resultant of this force and the force R, represented respectively by AC and CD, is represented by AD and acts through O. Thus, the forces P, Q, R are equivalent to a force represented by AD and acting through O.

Proceeding in this way, we see that the whole system is equivalent to a single force represented by AF and acting through O.

54. In the preceding article, in planning out the force diagram $ABCDEF$, the forces may be taken in any order, and the same result is obtained. The system cannot admit of two resultants different in magnitude, or in direction, or in line of action, since such resultants would have to be equivalent to one another, which is impossible by Art. 14.

55. Geometrical Condition of Equilibrium.—In Art. 53, if the final point F of the force diagram coincides with the initial point A, the resultant of the system vanishes. In such a case, replacing the forces P, Q, R, S by their resultant, we reduce the system to two forces both acting through O, namely—the resultant of P, Q, R, S, represented by AE, and the force T, represented by EA. These two forces are equal and act in opposite directions along the same straight line through O. Hence the forces are in equilibrium. Thus we have the proposition known as the **Polygon of Forces**:—

If any number of concurrent forces are represented in magnitude and direction by the sides of a polygon taken one way round, then the forces are in equilibrium.

This, of course, includes the Triangle of Forces as a particular case.

56. *Conversely*, if a number of concurrent forces are in equilibrium, and if straight lines AB, BC, CD, DE, EF are drawn to represent them, the force diagram commencing at the point A and terminating at the point F, then the final point F must coincide with the initial point A. For, otherwise, the system would be equivalent to a resultant represented by AF. Thus we have the following **Converse to the Polygon of Forces**:—

If a number of concurrent forces are in equilibrium, and if straight lines are drawn to represent them one after another, the arrows going one way round, then the final point of the force diagram coincides with the initial point.

The more general converse of the Polygon of Forces corre-

sponding to the converse of the Triangle of Forces considered in Art. 41 *is not true*, because two polygons with their sides respectively parallel are not necessarily similar.

57. Principle of Resolved Parts.—*The sum of the resolved parts in any direction of any number of concurrent forces is equal to the resolved part of their resultant in that direction.*

In the figure of Art. 53, let any straight line OX be drawn through O, and suppose that perpendiculars are drawn from A, B, C, D, E, F upon OX.

Then it is geometrically obvious that the sum of the projections upon OX of the lines AB, BC, CD, DE, EF is equal to the projection of AF upon OX, each projection having its own proper sign.

But the projection of any line represents the resolved part of the force represented by that line.

Therefore the sum of the resolved parts of the forces P, Q, R, S, T in the direction OX is equal to the resolved part of their resultant in that direction, each resolved part having its own proper sign.

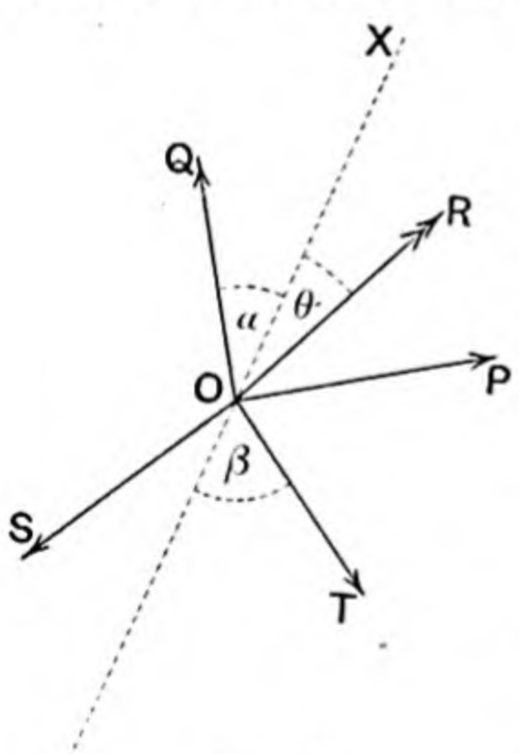

FIG. 45.

The student should notice the following particular cases:

(i) The sum of the resolved parts of the forces in the direction of their resultant is equal to the resultant.

(ii) The sum of the resolved parts of the forces in a direction perpendicular to their resultant is zero.

(iii) If the forces are in equilibrium, the sum of their resolved parts in every direction is zero.

Example.—If R is the resultant of the forces P, Q, S, T, indicated in Fig. 45, P and Q being at right angles, and S and T at right angles, we have, resolving along OX,

$$R \cos \theta = P \sin \alpha + Q \cos \alpha - S \sin \beta - T \cos \beta.$$

58. *If the sum of the resolved parts, in a certain direction, of any number of concurrent forces is zero, then either the forces are in equilibrium, or their resultant is at right angles to that direction.*

Let the sum of the resolved parts of the forces in the direction

OX be zero, and let the force diagram commence at A and finish at F.

Then the projection of AF upon OX is zero.

Hence, either F must coincide with A, or AF must be perpendicular to OX.

We have shown in Art. 55 that if F coincides with A, the forces are in equilibrium, and if F does not coincide with A, they are equivalent to a resultant force represented by AF.

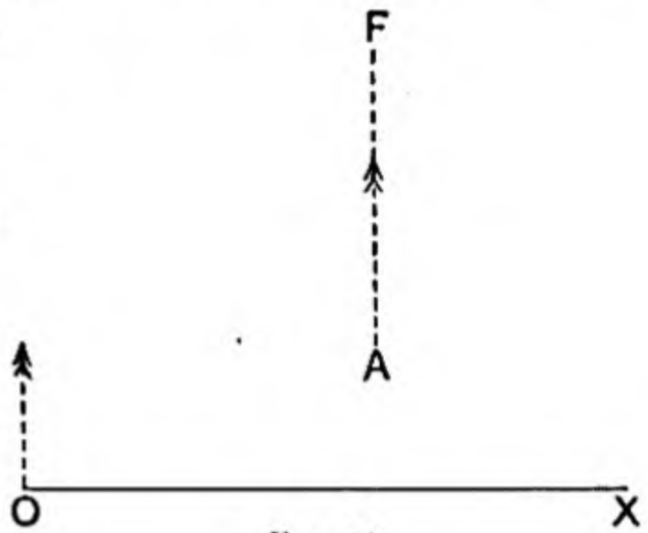

FIG. 46.

Thus, either the forces are in equilibrium, or their resultant is at right angles to OX.

59. Analytical Conditions of Equilibrium.—*If the sums of the resolved parts in two directions of any number of concurrent forces are separately zero, then the forces are in equilibrium.*

If the forces were equivalent to a resultant, it would, by the preceding article, be perpendicular to each of the two given directions, which is impossible. Hence the forces must be in equilibrium.

60. Calculation of Resultant.—*To find by calculation the magnitude and direction of the resultant of a given system of concurrent forces.*

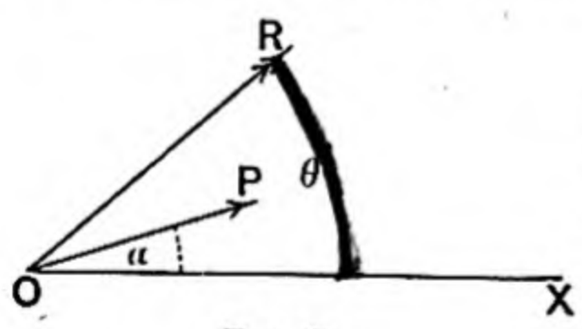

FIG. 47.

Let P be the measure of one of the forces, acting at an angle α with a known straight line OX drawn through the point O, where the lines of action of the forces intersect.

Suppose that the measure of the resultant is R, and that it makes an angle θ with OX.

Then, resolving along and perpendicular to OX, we have

$$\begin{cases} R\cos\theta = \text{sum of all such terms as } P\cos\alpha = \Sigma(P\cos\alpha) \\ R\sin\theta = \text{sum of all such terms as } P\sin\alpha = \Sigma(P\sin\alpha) \end{cases}$$

These two equations are sufficient to determine R and θ, as

Squaring and adding, we have

$$R^2 = [\Sigma(P\cos\alpha)]^2 + [\Sigma(P\sin\alpha)]^2$$

$$\therefore R = \sqrt{[\Sigma(P\cos\alpha)]^2 + [\Sigma(P\sin\alpha)]^2}$$

Also
$$\tan\theta = \frac{\Sigma(P\sin\alpha)}{\Sigma(P\cos\alpha)}.$$

Example.—In the case of the system indicated in Fig. 48, we have

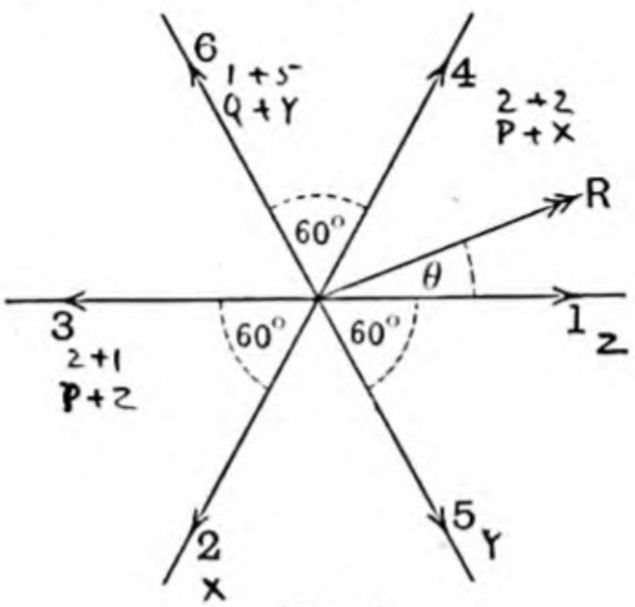

FIG. 48.

$$R\cos\theta = 1 + 4\cos 60^\circ - 6\cos 60^\circ - 3 - 2\cos 60^\circ + 5\cos 60^\circ$$
$$= -2 + \cos 60^\circ$$
$$= -\frac{3}{2}$$

Also, $R\sin\theta = 4\cos 30^\circ + 6\cos 30^\circ - 2\cos 30^\circ - 5\cos 30^\circ$
$$= 3\cos 30^\circ$$
$$= \frac{3\sqrt{3}}{2}$$

$$\therefore R^2 = \left(\frac{3}{2}\right)^2 (1+3) = 3^2$$
$$\therefore R = 3$$

Now $\cos\theta$ is negative, $\sin\theta$ is positive, and $\tan\theta = -\sqrt{3}$
$$\therefore \theta = 120^\circ.$$

Thus the resultant is of magnitude 3, and is in the same direction as the force 6.

Otherwise.—The forces 1 and 3 are equivalent to a force 2 in the direction of the greater; the forces 4 and 2, are equivalent to a force 2 in the direction of the greater; and the forces 6 and 5 are equivalent to a force 1 in the direction of the greater. Thus the given system is equivalent to the three forces indicated in Fig. 49.

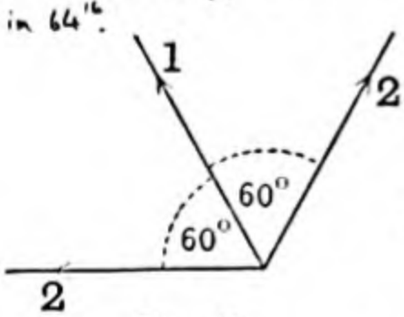

FIG. 49.

The two equal forces 2, acting at an angle of 120°, are equivalent to a force 2 in the direction of the middle force.

Thus the resultant is a force 3 in the direction of the original force 6.

EXAMPLES VII

1. If the side of a regular hexagon $ABCDEF$ represents a force of 100 pounds' weight, find the magnitudes of the forces represented by the straight lines AE, AD, FB; and supposing them to act at a point, determine the magnitude and direction of the resultant of the three forces.

2. A, B, C, D are the angular points of a parallelogram taken one way round; find the resultant of the forces represented by AB, AC, AD.

3. If a given straight line AB represents a force equal to the weight of 1 pound, construct a line which shall represent a force equal to the weight of $\sqrt{2}$ pounds.

$ABCD$ is a square. Forces of 1, $2\sqrt{2}$, 3, $\sqrt{2}$ pounds' weight act at a point in the directions of the lines AB, BD, DA, AC respectively; determine their resultant geometrically or otherwise.

4. $ABCD$ is a square. Forces of 1, 2, 3, $2\sqrt{2}$ pounds' weight act at a point in the directions of the lines AB, BC, CD, DB respectively; determine their resultant.

5. Find the resultant of forces 6, 2, 3 pounds' weight acting in one plane at a point, the middle one being inclined to each of the others at an angle of 60°.

6. Find the resultant of forces 5, 3, 4 pounds' weight acting in one plane at a point, the middle one being inclined to each of the others at an angle of 60°.

7. Find the resultant of forces 12, 9, 4 pounds' weight acting in one plane at a point, the angle between each pair being 120°.

8. The triangle ABC has its sides BC, CA, AB of lengths 4, 5, 6 inches respectively. Forces of 6, 5, 9 pounds' weight act at a point in the directions of the lines BC, CA, AB respectively; determine their resultant.

9. The triangle ABC has its sides BC, CA, AB of lengths 7, 9, 12 inches respectively. Forces of 7, 12, 16 pounds' weight act at a point in the directions of the lines BC, CA, AB respectively; determine their resultant.

10. Forces X, $P+X$, $Q+X$ act at a point in the directions of the sides of an equilateral triangle, taken one way round. Show that they are equivalent to two forces P and Q acting at an angle of 120°.

11. Forces P, Q, X act at a point in one plane, the middle one being inclined to each of the others at an angle of 60°. If the force P is removed, the resultant is of the same magnitude as before. Find X.

12. $ABCD$ is a square, and E is a point dividing BC in the ratio 3 : 1. Find the resultant of forces of 12, 5, 9 pounds' weight acting along AB, AE, AD respectively.

13. The length of a side of the regular hexagon $ABCDEF$ is a; forces represented in magnitude and direction by the lines AB, AC, AD, AE, AF act at A. Show that their resultant will be represented by a line of length $6a$.

14. Forces X, Y, Z, $P+X$, $Q+Y$, $P+Z$ act at a point in the directions of the sides of a regular hexagon, taken one way round.

Show that their resultant is a force $P+Q$ in the direction of the force $Q+Y$.

15. Forces 1, 2, 5, 6, 4, 3 act at a point in the directions of the sides of a regular hexagon, taken one way round. Find the force which will balance the system.

16. $ABCDEF$ is a regular hexagon. Find the resultant of forces of 7, 9, 5, 9, 7 pounds' weight acting along AB, CA, AD, AE, AF respectively.

17. $ABCDEF$ is a regular hexagon. Forces X, P, Y, Z, $Y+Q$ act along AB, AC, AD, AE, AF respectively. Find X in terms of P and Q, in order that the resultant may be along AE.

18. $ABCDEF$ is a regular hexagon. Forces of 2, X, 4, Y, 3 pounds' weight act along AB, CA, AD, AE, FA respectively. Determine the values of X and Y in order that the system may be in equilibrium.

19. ABC is an equilateral triangle, and D is the middle point of BC. Forces of 1, 5, X, Y pounds' weight act at A in the directions of the lines CB, AC, AB, DA respectively. Determine the values of X and Y in order that the system may be in equilibrium.

20. ABC is an equilateral triangle, and D is the middle point of BC. Find the resultant of forces of 3, 2, 4, 2 pounds' weight acting at A in the directions of the lines BC, DA, CA, AB respectively.

21. E and F are the middle points of the sides CA and AB respectively of an equilateral triangle ABC. Find the resultant of forces of 5, 2, 1, 3 pounds' weight acting at A in the directions of the lines CB, CA, BE, FC respectively.

22. E and F are the middle points of the sides CA and AB respectively of an equilateral triangle ABC. Forces 3, P, 3, 4, Q act at A in the directions of the lines BE, CA, CB, AB, FC respectively. Find the values of P and Q in order that the system may be in equilibrium.

23. E and F are the middle points of the sides CA and AB respectively of an equilateral triangle ABC. Forces P, X, Q, X, Y act at A in the directions of the lines BC, BE, CA, CF, AB respectively. If the system is in equilibrium, determine the values of X and Y in terms of P and Q.

24. OB is a diagonal of the square $OABC$; D and E are the middle points of AB and BC respectively. Find the resultant of forces P acting along OB, Q acting along OD, and Q acting along OE.

25. Let O be the position of a particle, and OA a straight line drawn through O. Find the magnitude and direction of the resultant of forces of 10, 18, 20, 16 pounds' weight acting on the particle, when their directions make with OA angles of 0°, 30°, 90°, 135° respectively, all measured in the same sense.

26. Forces of magnitudes 3, 4 and 5 act at a point O in directions lying in one plane, and making angles of 15°, 60° and 135° respectively with a line OA in the same plane. Find the magnitude of the resultant.

27. Forces of 3, 4, and 6 pounds' weight make angles of 90°, 60°, and 30° respectively with a force of 2 pounds' weight (the angles being measured in the same direction). Find the magnitude of the resultant, and the angle its direction makes with the force of 2 pounds' weight.

28. Five forces P, Q, X, R, Y act at O, the centre of a regular

pentagon $ABCDE$, in the directions OA, OB, OC, OD, OE respectively. If the system is in equilibrium and the forces P, Q, R are given, find the magnitudes of X and Y.

29. $ABCDE$ is a regular pentagon. Forces of n^2+2n, n, n, $2n+5$ pounds' weight act along the lines AB, AC, AD, AE respectively; show that their resultant is $\sqrt{n^4+5n^3+15n^2+2n+25}$ pounds' weight.

30. $ABCDE$ is a regular pentagon. Forces of 3, 24, 24, 16 pounds' weight act along the lines AB, AC, AD, EA respectively; show that their resultant is 41 pounds' weight.

61. Systems of Jointed Rods.—Suppose that a number of fine straight rods, of no appreciable weight, are jointed together at their extremities, so that each rod is in equilibrium under the influence of forces which act only at its extremities.

Let us consider the equilibrium of a single rod AB, jointed at A and B to the adjoining parts of the system. The only external forces that act upon it are those at A and those at B. Those at A are equivalent to a single force at A, and those at B to a single force at B. Hence, as in Art. 18, the resultant action at A must be equal and opposite to the resultant action at B, and both must act in the line AB, either outwards or inwards. Also the whole rod is in a state of direct tension or compression at every point throughout its length.

In the one case the rod is a *tie*, and might be replaced by a string, which would answer the purpose theoretically just as well. The effect is to bind together the two parts of the structure at A and B, the rod *pulling* the structure at each end in its own direction.

In the other case the rod is a *strut*, and its effect is to keep the two parts of the structure at A and B apart from one another, the rod *pushing* the structure at each end in its own direction.

If the rod is of appreciable weight, or if it has forces applied to it at points other than its extremities, then the above does not apply. We shall consider such cases in a future chapter.

62. If a number of fine light rods, as above, are jointed together at a common extremity O, we consider the equilibrium of the portion of the system included within a closed curve drawn round O, just as in Art. 45. The force diagram for this part of the system will consist of a polygon, whose sides are parallel to the rods which meet at O, and in which the arrows must go one way round. The direction of the arrow decides in each case, not already known, whether the rod is a *strut* or a *tie*.

This is briefly described as considering the equilibrium of O, and the force diagram is called the polygon of forces for the joint O.

For further discussion of the geometrical treatment of the equilibrium of systems of jointed rods, see the author's *Elementary Geometrical Statics*.

The analytical treatment of the equilibrium of any system of such jointed rods consists in resolving in each of two different directions for the equilibrium of each joint.

63. Ex. 9.—*A fine light rod AB, of length c, is capable of turning freely in a vertical plane about the point A, which is fixed. To the point B is attached a fine light string, supporting at its other extremity a mass of weight W. Another fine light string, of length a, connects B with the fixed point C, situated at a distance b vertically above A. Find the tensions of the strings, the stress in the rod, and the action at A.*

In the position of equilibrium, the two strings are straight and the mass rests vertically below B.

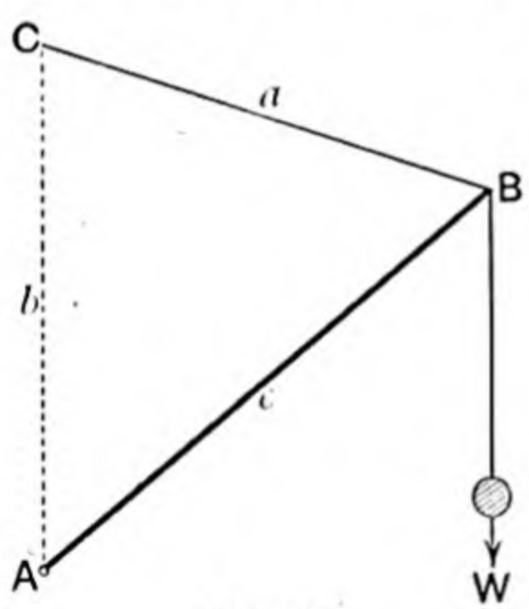

FIG. 50.

The rod is at rest under forces acting only at A and B. Therefore the stress in the rod is at every point in its own direction, and the action at A is in the line AB.

The triangle ABC has its sides parallel to the forces which act upon the portion of the system at B; also the tension of the vertical string we see at once is W. Hence, on the scale in which CA represents W, AB and BC represent the thrust of the rod and the tension of the string BC.

Thus, the thrust of the rod is $\frac{c}{b}W$, the tension of BC is $\frac{a}{b}W$;

and the action at A is $\frac{c}{b}W$ in direction AB.

***64.** Ex. 10.—*Four fine light rods are smoothly jointed at their extremities to form a quadrilateral framework $OBAC$, the opposite joints being connected by two fine light strings OA and BC in a state of tension. From any point a in BC, or BC produced either*

way, a straight line is drawn parallel to OB to meet OA in γ; from γ a straight line is drawn parallel to AB to meet BC in o; from o a straight line is drawn parallel to AC to meet OA in β; and the straight line βα is drawn. It is required to prove that βα is parallel to OC, and that the thrusts in the rods OB, BA, AC, CO and the tensions of the strings OA, BC are proportional to γα, oγ, oβ, αβ, βγ, oα respectively.

Let the scale of representation of force be so chosen that $o\alpha$ may represent the tension of the string *BC*. Then $o\alpha\gamma o$ (this way round) is the force diagram for the joint *B*, so that the lines $\alpha\gamma$ and γo represent the thrusts of the rods *OB* and *AB* upon the joint *B*. Hence $o\gamma$ represents the thrust of the rod *AB* upon the joint *A*. Thus the force diagram for the joint *A* is $o\gamma\beta o$, so that βo represents the thrust of the rod *CA* upon the joint *A*, and $\gamma\beta$ represents the tension of the string *OA*.

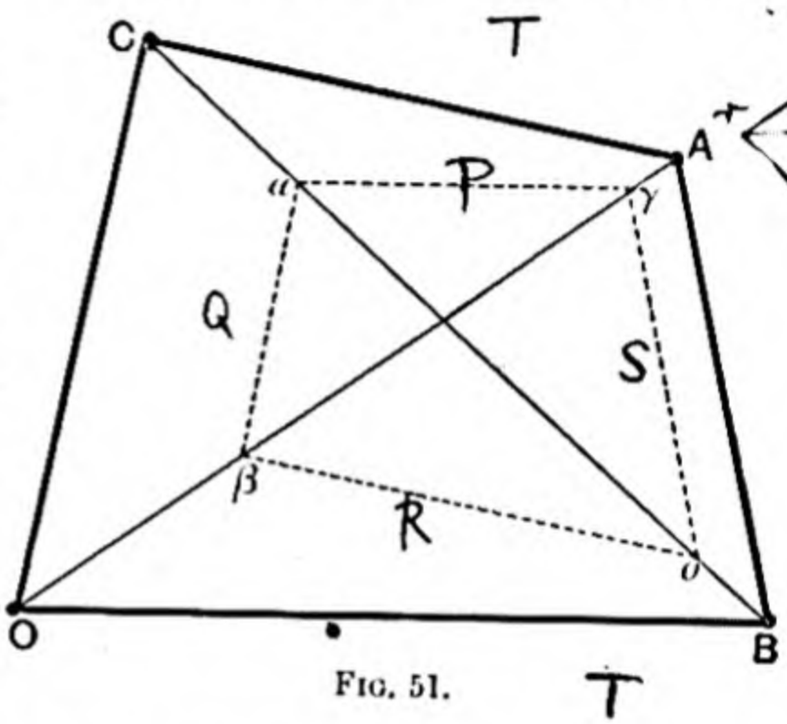

FIG. 51.

Hence, in the triangle $\alpha o\beta$, αo represents the pull of the string *BC* at *C*, and $o\beta$ represents the thrust of the rod *AC* upon the joint *C*; therefore $\alpha o\beta\alpha$ must be the force diagram for the joint *C*, so that $\beta\alpha$ is parallel to *OC* and represents the thrust in the rod *OC*.

Thus, the thrusts in the rods *OB*, *BA*, *AC*, *CO*, and the tensions of the strings *OA*, *BC* are proportional to $\gamma\alpha$, $o\gamma$, $o\beta$, $\alpha\beta$, $\beta\gamma$, $o\alpha$ respectively.

***65.** Ex. 11.—*A number of fine light rods are freely jointed at their extremities so as to form a plane figure, each joint connecting two rods, and the whole is in equilibrium under the influence of forces applied at the joints in the same plane. It is required to consider the conditions of equilibrium, and to determine the stresses in the rods.*

(i) *Geometrically.*—Fig. 52 represents four of the rods, marked *a*, *b*, *c*, *d*. Let the joints be called *ab*, *bc*, *cd* respectively.

Let *AB*, *BC*, *CD* be drawn to represent the forces applied at

the joints ab, bc, cd respectively. Then we shall show that the straight lines drawn through A, B, C, D parallel to the rods a, b, c, d respectively must be concurrent.

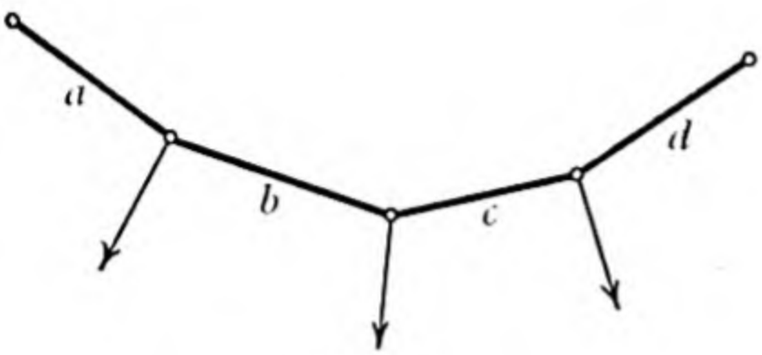

FIG. 52.

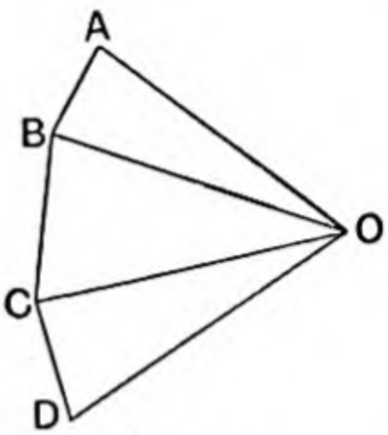

FIG. 52a.

Draw AO and BO, parallel to the rods a and b respectively, to meet in O. Then $ABOA$ (this way round) is the triangle of forces for the joint ab. Thus BO represents the action of the rod b upon the joint ab, and therefore OB represents the action of the same rod upon the joint bc. Hence, joining OC, we see that $BCOB$ is the triangle of forces for the joint bc; so that CO is parallel to the rod c, and represents its action upon the joint bc. Similarly DO is parallel to the rod d, and represents its action upon the joint cd.

Thus the straight lines drawn through A, B, C, D parallel to the rods a, b, c, d respectively must be concurrent in O, and the straight lines so drawn represent the stresses in the rods.

(ii) *Analytically.*—We may either resolve twice or use Lamé's Theorem for the equilibrium of each joint.

Particular Example.—The fine light rod b, having two masses attached to its extremities, one of known weight W and the other of unknown weight X, is supported by two fine light strings a and c. Suppose that it is required to determine the value of X, in order that the rod may rest in a horizontal position, the string a being inclined at a known angle α to the vertical, and the string c being at right angles to the string a.

(i) *Geometrically.*—Draw AB vertically downwards to represent W, AO and BO parallel to the string a and the rod b respectively to meet in O, and OC parallel to the string c, to meet AB produced in C. Then BC represents X.

Also, evidently,

$$\begin{cases} AO = W \sec \alpha \\ BO = W \tan \alpha \\ CO = BO \sec \alpha = W \tan \alpha \sec \alpha \\ BC = BO \tan \alpha = W \tan^2 \alpha. \end{cases}$$

Thus X is of magnitude $W\tan^2\alpha$, the tensions of the strings a and c are $W\sec\alpha$ and $W\tan\alpha\sec\alpha$ respectively, and the thrust in the rod is of magnitude $W\tan\alpha$.

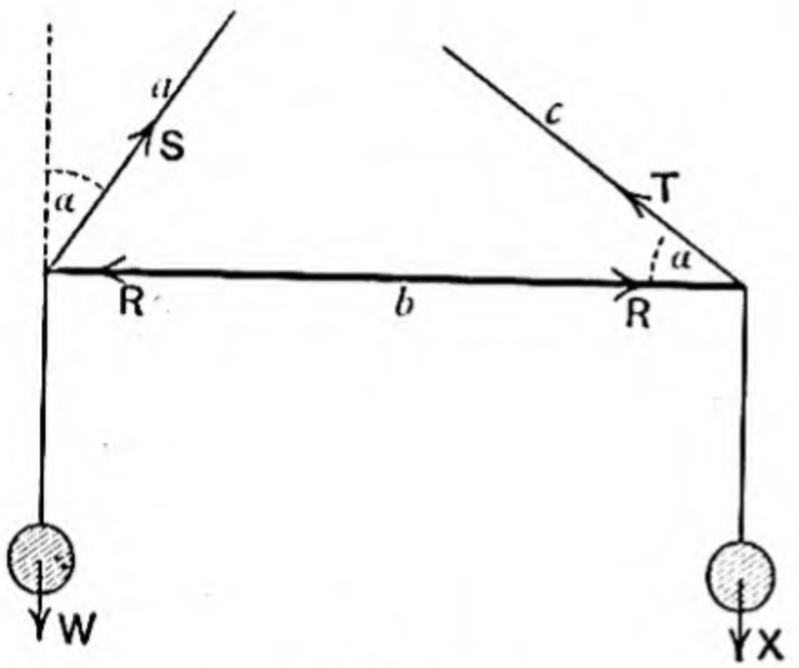

FIG. 53.

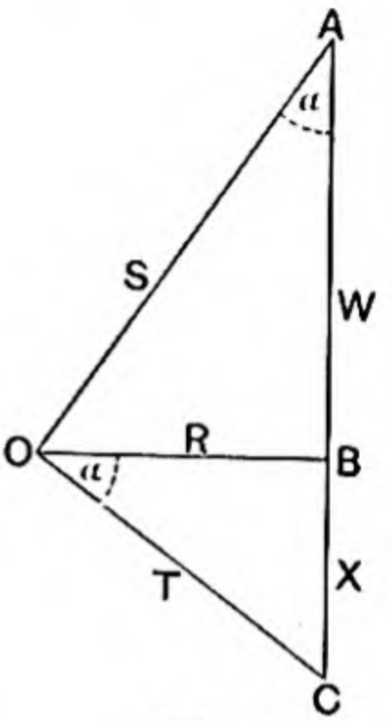

FIG. 53a.

(ii) *Analytically.*—Let S and T be the tensions of the strings a and c respectively, and R the thrust in the rod.

Then, resolving vertically and horizontally for the equilibrium of the portions of matter in the immediate neighbourhood of each end of the rod, we have

$$\begin{cases} S\cos\alpha = W \\ R = S\sin\alpha \\ T\sin\alpha = X \\ R = T\cos\alpha. \end{cases}$$

These equations give the same results as before.

***66.** Ex. 12.—*Four fine light rods AB, BC, CD, DA of lengths a, b, a, b respectively, b being greater than a, are freely jointed together at their extremities to form a rectangular framework, which is stiffened by another fine light rod connecting the hinges B and D. The framework is capable of turning freely in a vertical plane about the point A which is fixed, and rests, with BD horizontal, under a load of given weight W applied at C, and an unknown horizontal force applied at D. Find the magnitude of the force at D, and the thrust in the rod BD.*

[For a geometrical solution see Art. 140 of the author's *Elementary Geometrical Statics.*]

Let $\widehat{ABD} = \alpha$, so that $\tan\alpha = \dfrac{b}{a}$.

Let P be the value of the unknown force applied at D, S the

thrust in the rod BD, T and T' the tensions of BC and CD respectively.

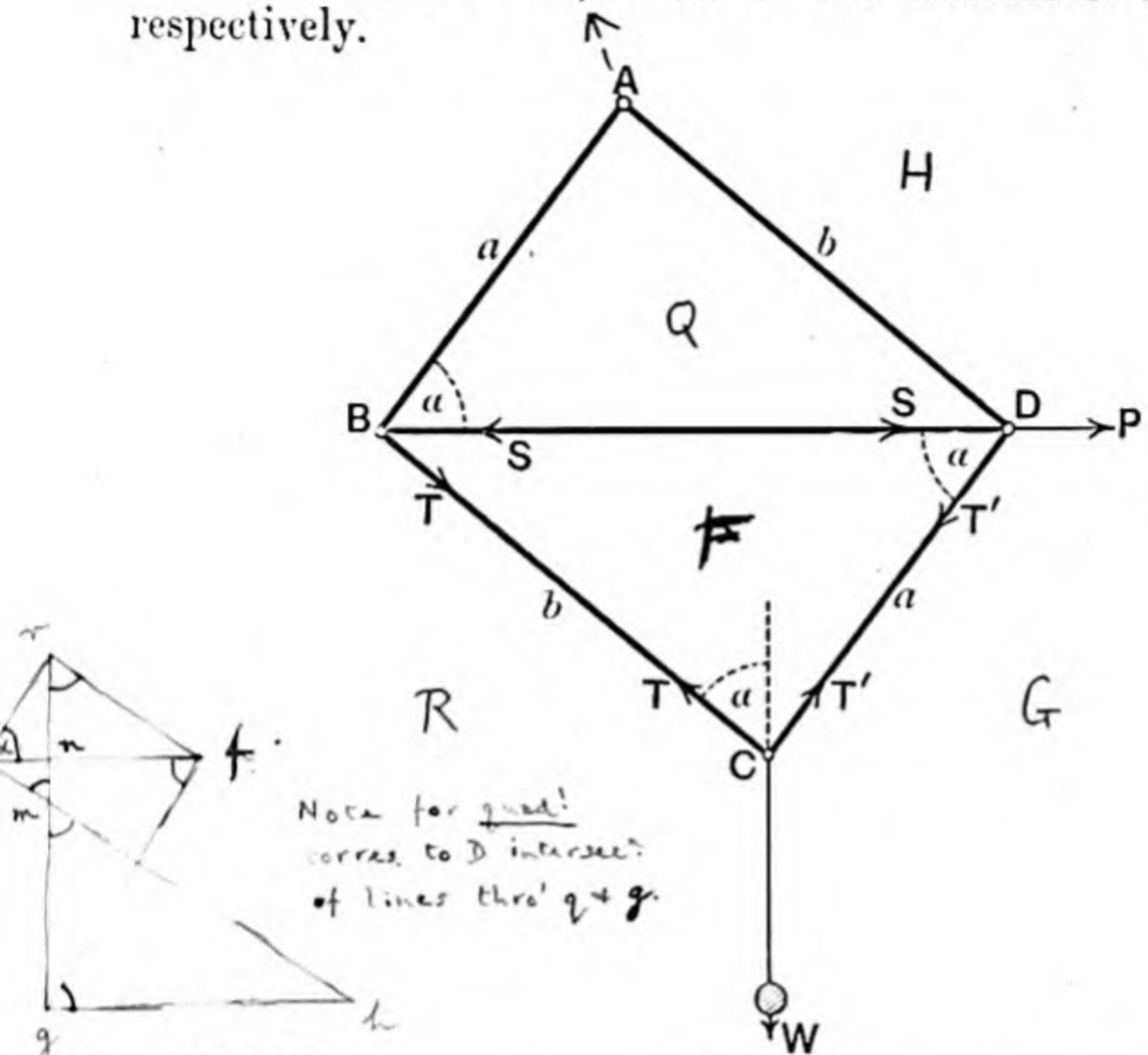

FIG. 54.

Resolving along BC and CD for the equilibrium of the joint C, we have

$$\begin{cases} T = W\cos a, \\ T' = W\sin a. \end{cases}$$

Now consider the equilibrium of the joint B. In order to eliminate the action in the rod AB, we resolve perpendicular to that rod, and obtain

$$\begin{aligned} S\sin a &= T' \\ &= W\cos a, \end{aligned}$$

$$\therefore\ S = W\cot a = W\frac{a}{b}.$$

Also, resolving perpendicular to AD for the equilibrium of the joint D, we have

$$\begin{aligned} (P+S)\cos a &= T' \\ &= W\sin a, \end{aligned}$$

$$\begin{aligned} \therefore\ P &= W\tan a - S \\ &= W(\tan a - \cot a) \\ &= W\left(\frac{b}{a} - \frac{a}{b}\right). \end{aligned}$$

*67. Ex. 13. **Suspension Bridge.**—In the suspension bridge represented in Fig. 55, we suppose that the chain and the vertical suspending rods are of no appreciable weight, and that the latter are sufficiently numerous to enable us to look upon the chain as forming a continuous curve.

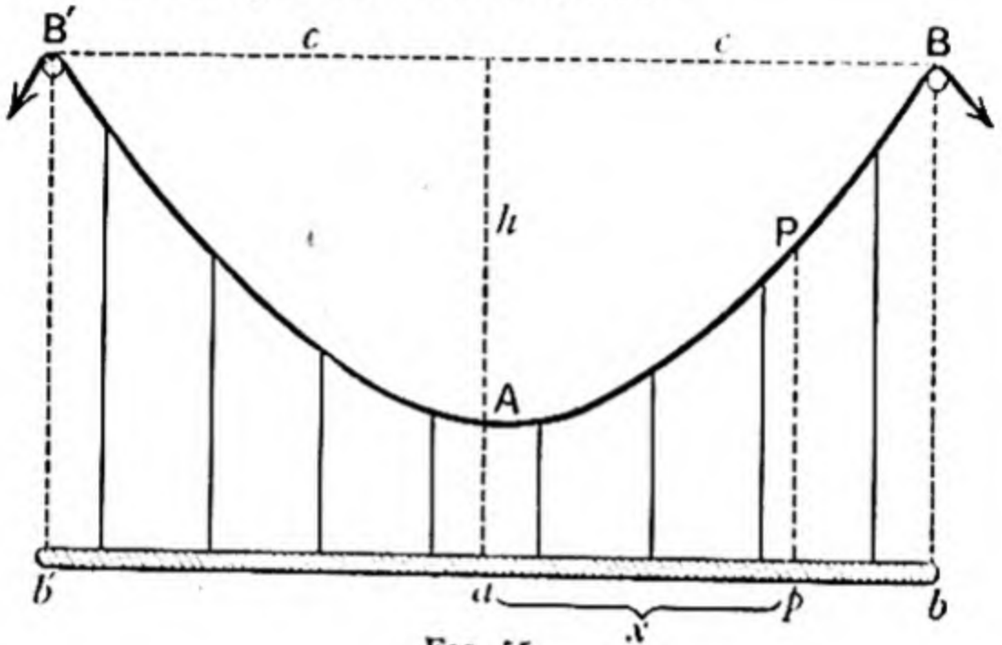

FIG. 55.

Let the span of the bridge be of length $2c$, and let W be the total weight of the roadway. Let A be the lowest point and P any other point of the chain, T_0 the tension at A and T the tension at P, and let ϕ be the inclination of the chain at P to the horizontal.

Let us consider the equilibrium of the portion AP. We assume that the total load supported by this part of the chain is the portion ap of the roadway vertically below AP, and that the resultant weight of this portion acts vertically downwards along a line midway between Aa and Pp.

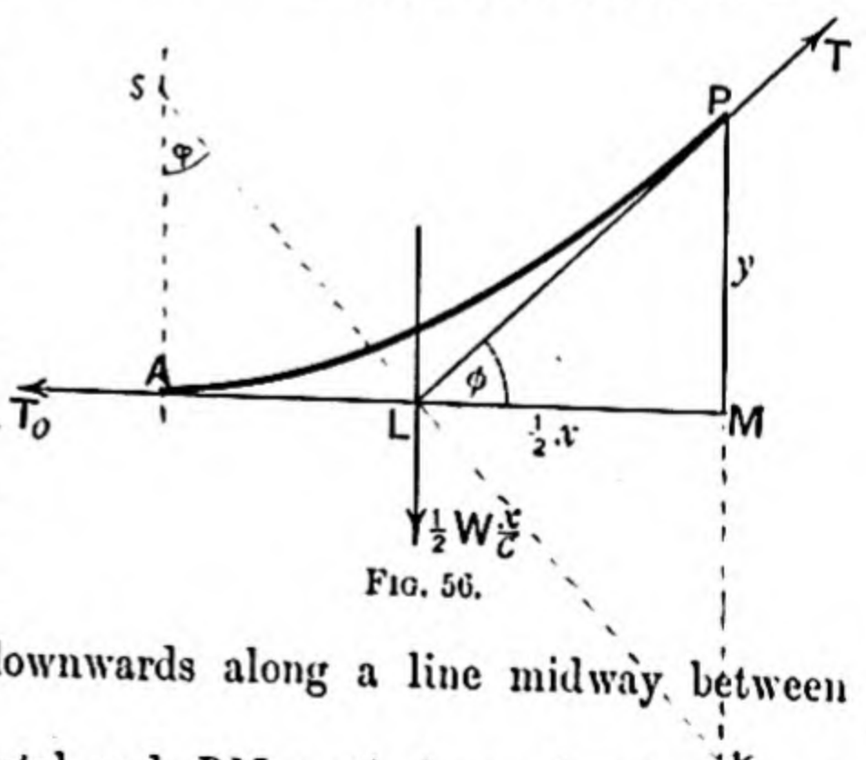

FIG. 56.

Draw AM horizontal and PM vertical, meeting in M, and let AM and MP be of lengths x and y respectively.

The total load upon AP is of weight $W\frac{x}{2c}$, acting vertically downwards through L, the middle point of AM.

Thus the forces acting externally upon AP are equivalent to—

$$\begin{cases} \text{At } A \text{ a force } T_0 \text{ in a horizontal direction.} \\ \text{At } P \text{ a force } T \text{ at an angle } \phi \text{ with the horizontal.} \\ \text{Through } L \text{ a force } \frac{1}{2}W\frac{x}{c} \text{ vertically downwards.} \end{cases}$$

The lines of action of these three forces must meet at L. Hence the tangent at P bisects the line AM.

$$\therefore \tan\phi = \frac{MP}{LM} = \frac{y}{\frac{1}{2}x} = \frac{2y}{x}.$$

Resolving vertically and horizontally, we have

$$\begin{cases} T\sin\phi = \frac{1}{2}W\frac{x}{c} \\ T\cos\phi = T_0. \end{cases}$$

Therefore, eliminating T,

$$\frac{Wx}{2T_0c} = \tan\phi = \frac{2y}{x},$$

so that

$$y = \frac{W}{4T_0c}x^2.$$

Thus, for different points P, y varies as x^2.

The student of Higher Mathematics will see that the curve of the chain is a parabola with its vertex at A and its axis vertically upwards.

For any given value of y there are two equal and opposite values of x. Thus the curve is symmetrical about the vertical through A, which must therefore lie vertically below the middle point of the horizontal line BB'.

Let h be the depth of A below BB'; then when $x = c$, $y = h$.

$$\therefore h = \frac{W}{4T_0c}\cdot c^2,$$

$$\therefore T_0 = \frac{c}{4h}W,$$

and the equation connecting x and y becomes

$$y = h\left(\frac{x}{c}\right)^2.$$

Also

$$\tan\phi = \frac{2y}{x} = 2\left(\frac{h}{c}\right)\left(\frac{x}{c}\right).$$

And

$$T = T_0 \sec\phi = \frac{c}{4h} W \sqrt{1+\tan^2\phi}$$

$$= \frac{1}{4} W \sqrt{\left(\frac{c}{h}\right)^2 + 4\left(\frac{x}{c}\right)^2}.$$

*EXAMPLES VIII

1. Four forces, acting at a point, are represented by AB, BC, DC, AD; show that their resultant is represented by twice AC.

2. Prove that four *equal* forces, acting at a point in one plane, cannot be in equilibrium unless they consist of two pairs of equal and opposite forces.

3. Two equal forces P, acting at an angle 2α, are in equilibrium with two equal forces Q, acting at an angle 2β, all four being in one plane and acting at the same point. Prove that *either* α and β are each equal to $\frac{\pi}{2}$, *or* $P\cos\alpha = Q\cos\beta$, and in the second case the bisector of the angle between the two forces P must be in the same straight line as the bisector of the angle between the two forces Q.

4. Four forces in equilibrium act at the same point and are represented in magnitude and direction by AB, CD, AD, CB. Show that A, B, C, D must be the angular points of a parallelogram.

5. A, B, C, D are any four points situated in one plane. Find a point O such that forces represented by OA, OB, OC, OD may be in equilibrium.

Hence prove that the straight line joining the middle points of AB and CD, the straight line joining the middle points of BC and DA, and the straight line joining the middle points of AC and BD are concurrent and bisect one another.

6. A system of coplanar forces, acting at a point O, are in equilibrium, one of them being of magnitude P in direction OA. If the force P is turned through an angle θ, show that the system is then equivalent to a resultant of magnitude $2P\sin\frac{\theta}{2}$ in a direction inclined at an angle $\frac{\pi}{2}+\frac{\theta}{2}$ with OA.

7. Extend Art. 30 so as to include the case of any number of forces acting at a point.

8. $OABC$ is a parallelogram, and D is the middle point of AB. Find the resultant of forces represented by OA, $2(OD)$, OB, OC, all acting upon a particle at O.

9. Two fine light rods a and b are freely jointed to one another at a common extremity C, and to two fixed points at their other extremities. A given force is applied in a given direction to the joint C. Show how to determine the stresses in the rods, and whether they are ties or struts.

10. A fine light rod AB is capable of turning freely in a vertical

plane about its lower end A, which is fixed. A mass of weight W is suspended from B, and the whole is supported, with the rod inclined at an angle α to the vertical, by means of a fine horizontal string attached at B. Find the tension of the string, the thrust in the rod, and the action at A.

If the string cannot bear a tension greater than a certain value P, find the weight of the greatest mass which can be suspended from B without breaking the string.

Also, if there is no load supported at B, find in what direction a given force Q must be applied at B, in order to be just on the point of breaking the string.

11. A fine light rod AB, of length one foot, is capable of turning freely in a vertical plane about the point A, which is fixed. To the point B is attached a fine light string, supporting at its other extremity a mass of weight W. Another fine light string, of length 16 inches, connects B with a fixed point C, situated 8 inches vertically above A. Find the tension of BC and the thrust in AB.

If the rod cannot bear a thrust greater than a certain given value P, and if the string cannot bear a tension greater than $\frac{2}{3}P$, and if there is no load supported at B, determine the magnitude and direction of the force which, applied at B, will be on the point of breaking both rod and string simultaneously.

Find also the weight of the greatest mass which can be suspended from B.

12. Four fine light rods, each of length c, are smoothly hinged together at their extremities to form a rhombus $OBAC$, and the hinges at B and C are connected by another fine light rod, also of length c. If the whole is supported at O and a mass of weight W is suspended from A, find the thrust in BC.

13. In the preceding example, if the rod BC is of length a, find the thrust in BC.

14. Four fine light rods are freely jointed together at their extremities to form a quadrilateral framework $OBAC$, which is stiffened by another fine light rod connecting the opposite joints B and C. A mass of weight W is attached to the joint A, and the whole is suspended from O. Prove that

(i) If the angles OBA and OCA are both right, then the thrust in the rod BC is of magnitude W cosec BOC.

(ii) If the angles BOC and BAC are both right, then the thrust in the rod BC is of magnitude $W \sin OBA$.

(iii) If the angles OBC and OCB are each α, and the angles ABC and ACB each β, then the thrust in BC is of magnitude

$$\tfrac{1}{2}W(\cot\alpha + \cot\beta).$$

(iv) If the angles OBC and ABC are each α, and the angles OCB and ACB each β, then the thrust in BC is of magnitude

$$\left(\frac{2}{\tan\alpha + \tan\beta}\right)W.$$

15. In Example 14, if OB and OC are each of length 5 feet, AB and AC each of length 6 feet, and BC of length 8 feet, find the thrust in BC.

16. In Example 14, if BO and BA are each of length 3 feet, CO and CA each of length 2 feet, and BC of length 4 feet, find the thrust in BC.

17. Four fine light rods, of lengths b, c, b, c are smoothly hinged together at their extremities to form a parallelogram $OBAC$, which is stiffened by another fine light rod, of length a, connecting the opposite hinges B and C. A mass of weight W is attached to the hinge A, and the whole is suspended from O. Prove that the thrust in BC is of magnitude

$$\left\{2\left(\frac{b^2+c^2}{a^2}\right)-1\right\}^{-\frac{1}{2}}W.$$

18. In Art. 64, let a, b, c, a_1, b_1, c_1 be the lengths of the lines BC, CA, AB, OA, OB, OC respectively; also, let Q_1, R, Q, R_1, be the thrusts in the rods OB, BA, AC, CO respectively, and P_1, P the tensions of the strings OA, BC respectively.

Prove that

(i) If the framework is a parallelogram, then the thrusts of the rods and the tensions of the strings are proportional to their lengths.

(ii) If a circle can be circumscribed to the quadrilateral, then the thrusts of the rods and the tensions of the strings are proportional to the lengths of the opposite rods and strings respectively.

(iii) If a_1 bisects a, then $Q : R = b : c$ and $Q_1 : R_1 = b_1 : c_1$.

(iv) If b is parallel to b_1, then

$$P : Q : R : P_1 : Q_1 : R_1 = a : b_1 : c : a_1 : b : c_1.$$

(v) If $R : R_1 = c : c_1$, then b is parallel to b_1.

19. Three fine light rods, of lengths a, b, c, are smoothly hinged together at their extremities to form a triangular framework ABC, which is in equilibrium under the influence of forces P_1, Q_1, R_1 applied at A, B, C respectively along lines which intersect at any point O in the plane of the framework. The stresses in the rods a, b, c are of magnitudes P, Q, R respectively, and the lines OA, OB, OC are of lengths a_1, b_1, c_1 respectively. Prove that, if a geometrical construction is made similar to that of Art. 64, then

$$P : Q : R : P_1 : Q_1 : R_1 = o\alpha : o\beta : o\gamma : \beta\gamma : \gamma\alpha : \alpha\beta.$$

Hence show that

(i) If O is any point on the circum-circle of ABC, then

$$P : Q : R : P_1 : Q_1 : R_1 = a_1 : b_1 : c_1 : a : b : c.$$

(ii) If O is the orthocentre of ABC, then

$$P : Q : R : P_1 : Q_1 : R_1 = a_1 : b_1 : c_1 : a : b : c.$$

(iii) If O is the centre of any one of the four circles which touch the sides of the triangle ABC, then $P = Q = R$.

(iv) If O is the centroid of ABC, then

$$P : Q : R : P_1 : Q_1 : R_1 = a : b : c : 3a_1 : 3b_1 : 3c_1.$$

(v) If $OBAC$ is a parallelogram, then

$$P : Q : R : P_1 : Q_1 : R_1 = a : b : c : \sqrt{2(b^2+c^2)-a^2} : b : c.$$

20. Three fine light rods are freely jointed together at their extremities to form a triangular framework ABC, which is capable of turning freely in a vertical plane about the joint B, which is fixed.

A mass of weight W is suspended from C, and the framework is sustained in a position in which BC is horizontal, and A uppermost, by means of a horizontal force applied at A. Determine in the following cases the magnitude of the applied force, the stresses in the rods, and the reaction at B.

(i) The angle $ACB = \alpha$ and ABC is a right angle.
(ii) The angle $ACB = \alpha$ and BAC is a right angle.
(iii) The angles ABC and ACB are each α.
(iv) The angles BAC and ACB are each α.
(v) The rods BC, CA, AB are of lengths 8, 5, 7 feet respectively.

21. Four fine light rods are freely jointed at their extremities to form a parallelogram $OBAC$. A mass of weight W is attached at A, and the whole is supported at the opposite joint O. What horizontal forces must be applied at B and C, in order that the framework may rest with A vertically below O, and with OB and OC inclined at angles β and γ respectively to the vertical?

22. Four fine light rods, of lengths 13, 9, 13, 9 inches, are freely jointed at their extremities to form a parallelogram $OBAC$. The joint O is fixed, and the joint A is connected with a fixed point D by means of a fine elastic string, so that OAD is a straight line. The joints B and C are pulled apart 10 inches by forces, each of magnitude P, applied at B and C in the line BC. Find the tension of the string AD.

23. A light rod AA_1, of length $2a + b$, rests in a horizontal position with masses each of weight W suspended from A and A_1. A fine light string ABB_1A_1 has its extremities attached at A and A_1, and the whole is supported by means of two vertical forces applied at B and B_1. The portions of string AB, BB_1, B_1A_1 are of lengths c, b, c respectively, and the whole rests in a symmetrical position with BB_1 horizontal. Find the tensions of the different parts of the string.

24. Two fine light rods CA and AB, of lengths b and c respectively, are freely jointed together at A, which is fixed. The rods rest in a horizontal position, with masses of weight W and W' suspended from B and C respectively, being supported by two fine light strings connecting B and C with a fixed point O, situated at a distance a vertically above A. Determine the constraints at A and O.

25. A fine light string $ABCB_1A_1$, of length $3a$, has its extremities attached to two fixed points A and A_1, situated in a horizontal line, and supports at its middle point C a mass of weight W. Two fine light rods OB and OB_1, each of length a, are capable of turning freely about O, the middle point of AA_1, and are attached to the string at points B and B_1, each at a distance a from C. The strings AB and A_1B_1 being both vertical, determine the tensions of the different parts of the string, the thrusts in the rods, and the reaction at O.

26. Four fine light rods are freely jointed together at their extremities to form a quadrilateral framework $OBAC$, and a fine light string connects the hinges O and A. Masses of weight W and W' are suspended from B and C respectively, and the whole is supported at O. If OA is vertically downwards in the position of equilibrium, prove that

$$W : W' = CD : DB,$$

where D is the point of intersection of BC and AO.

Prove also that if T is the tension of the string OA, then

$$T : W + W' = AD : AO.$$

27. In the preceding example, if W is supposed known, determine W' and T in each of the following cases.

(i) $OBAC$ is a parallelogram.

(ii) OC is parallel to BA; and BD and CD are of lengths b and c respectively.

(iii) The angles BAO and CAO are each α; and the angles BOA and COA are each β.

(iv) The angles BAO and BOA are each α; and the angles CAO and COA are each β.

28. Four fine light rods AB, BC, CD, DA, of lengths 3, 4, 3, 4 feet respectively, are freely jointed at their extremities to form a rectangular framework, which is stiffened by another fine light rod connecting the hinges B and D. The framework is capable of turning freely in a vertical plane about the point A, which is fixed, and rests with DB vertically downwards, under a load of weight W applied at B and a vertical force at C. Find the magnitude of the force at C and the stresses in the different rods.

29. Four fine light rods, of equal length, are freely jointed at their extremities to form a rhombus $OBAC$, which is stiffened by another fine light rod OA. The framework rests with OA vertically downwards, a mass of weight W being suspended from A, and is supported by two forces applied at B and C in directions perpendicular to OB and OC respectively. Find the magnitudes of these forces and the stress in the rod OA, the sides of the rhombus being each inclined to the vertical at an angle α.

30. In Art. 67 a straight line SLK is drawn perpendicular to LP to meet the verticals through A and P in S and K respectively. Prove that

(i) S is a fixed point.

(ii) The locus of K is a fixed horizontal straight line.

(iii) $SP = PK$.

(iv) The tension at P is equal to the weight of a portion of the roadway equal in length to SK.

31. In a horizontal suspension bridge, of length a and weight W, the longest of the suspending rods are of length $\frac{1}{4}a$, and the tensions at the highest points of the chain are $\frac{5}{4}$ times the tension at the lowest point. Determine

(i) The length of the shortest suspending rod.

(ii) The tension at the lowest point of the chain.

(iii) The length of the suspending rod at a point one-quarter of the way across the bridge.

(iv) The inclination of the chain to the horizontal at that point.

(v) The tension of the chain at that point.

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CHAPTER V

EQUILIBRIUM ON A SMOOTH SURFACE

68. Equilibrium of a Particle in Contact with a Smooth Surface.—Let a particle be in contact with a smooth surface at O; then the surface offers a resistance R to any tendency which the particle may have to penetrate it. This resistance is a self-adjusting force and accommodates itself so as to balance, if possible, all the other forces that act upon the particle. If the surface is *smooth*, it cannot offer any resistance to a tendency which the particle may have to *slip* over it; in such a case the surface can offer a resistance only in the direction of the *normal* ON drawn away from it at O.

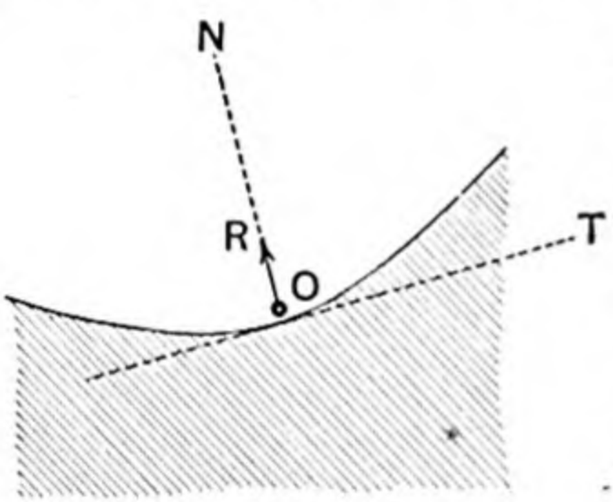


FIG. 57.

Suppose that a given system of forces is applied to the particle in a plane containing ON. Then it is necessary and sufficient for equilibrium that the resultant of the given system should be along NO. The surface then offers a resistance R in direction ON of sufficient magnitude to preserve equilibrium. The given system of forces and the force R form a system in equilibrium as for a particle free to move.

Analytical Conditions of Equilibrium.—Excluding the force R, the sum of the resolved parts of the given system of forces estimated in the direction of the tangent OT to the surface at O must be zero, while the sum of the resolved parts of the forces in the direction of the normal *NO towards the surface* must not be negative.

The second of these two conditions gives the value of the force R, and is necessary in order to ensure that the forces do not tend to move the particle away from the surface altogether. In the case of a bead threaded upon a fine smooth wire, or of a particle placed inside a fine smooth tube, this second condition is unnecessary.

Example.—Fig. 58 represents a circular hoop, fixed with the diameter AB vertically downwards. A smooth ring, of weight W, rests at C, being supported by a fine light string AC. Suppose that it is required to determine the tension of the string, when a given force F is applied to the ring in a direction perpendicular to AC.

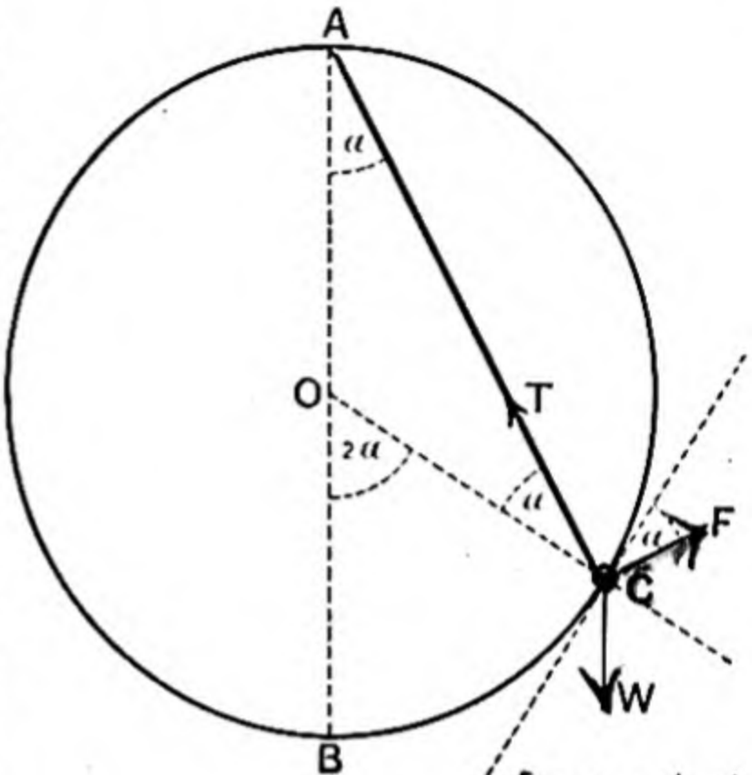


FIG. 58.

If O is the centre of the hoop, the angles OAC and OCA are equal. Let each be α. Then $\widehat{BOC}=2\alpha$, and the force F is inclined at an angle α to the tangent at C. Resolving along the tangent, we have

$$T\sin\alpha+F\cos\alpha=W\sin 2\alpha,$$
$$\therefore\ T=2W\cos\alpha-F\cot\alpha.$$

69. Equilibrium of a Heavy Particle on a Smooth Inclined Plane.—*A particle, of given weight W, is placed at O on a smooth plane inclined at a given angle α to the horizontal AC, and is sustained by some force P applied to it in some direction OL, which is in a vertical plane with AOB, the line of greatest slope of the inclined plane. It is required to examine the conditions of equilibrium, and in any given case of equilibrium to determine the pressure between the particle and the plane.*

Draw $M'OM$ vertically downwards, and $N'ON$ normal to the plane. The force P has to balance a known force W along OM, and a self-adjusting force R along ON. Hence OL must lie between OM' and ON'.

Draw DE vertically downwards to represent W, and EH of unlimited length parallel to ON, so that the angle $DEH=\alpha$.

The triangle of forces for the particle will be $DEFD$, where

F is a point in EH and FD parallel to OL. Thus, the different values of P, corresponding to different directions of OL, are

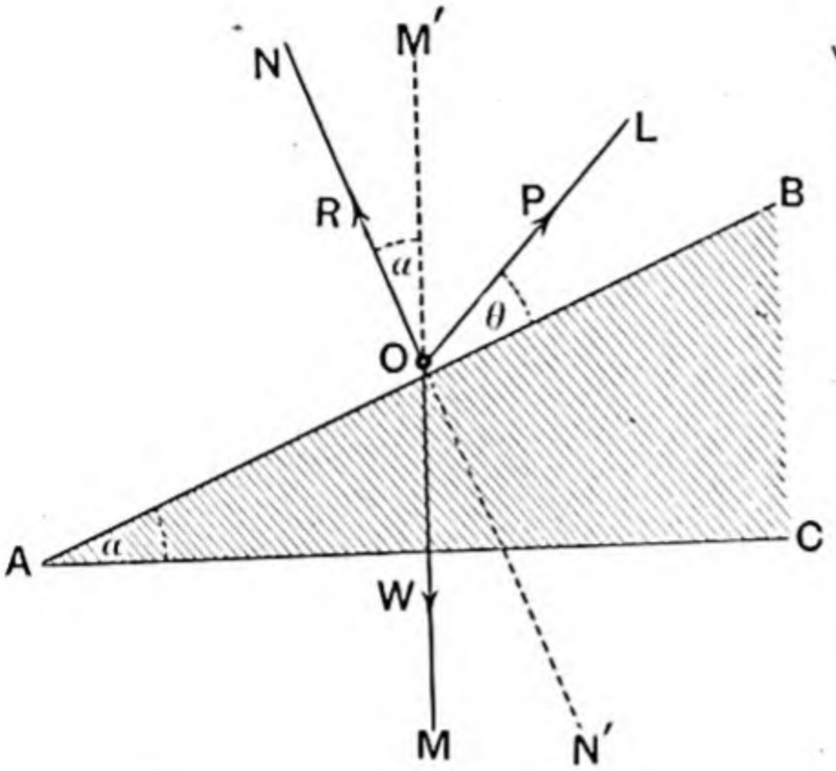


FIG. 59.

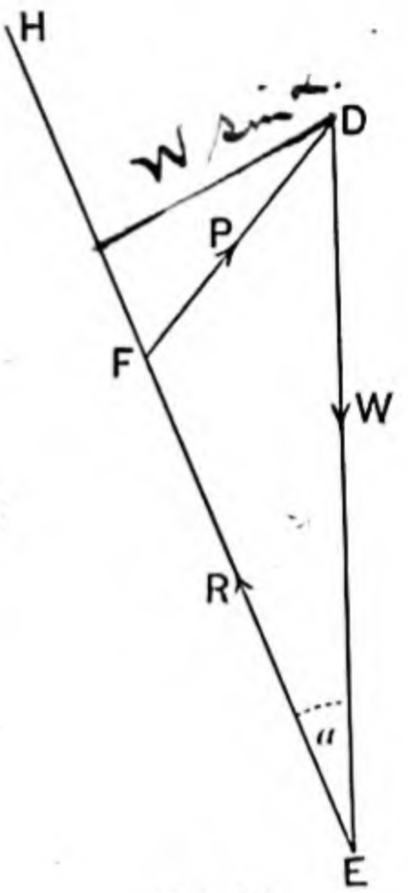


FIG. 59a.

represented by the different lines drawn to D from points in EH.

The shortest of these lines is that which is perpendicular to EH, and its length is evidently $W \sin \alpha$. Hence, the smallest force necessary to sustain the particle is a force $W \sin \alpha$ acting along OB.

For any given direction of OL, we draw DF parallel to LO, to meet EH in F. Then FD represents P, and EF represents R.

Analytically.—Let angle $BOL = \theta$. Then, resolving along and perpendicular to the plane, we must have

$$\begin{cases} P \cos \theta = W \sin \alpha \quad . \quad . \quad . \quad . \quad . \quad \text{(i)} \\ \text{and } (W \cos \alpha - P \sin \theta) \text{ must be positive,} \end{cases}$$

the expression $(W \cos \alpha - P \sin \theta)$ giving the value of the pressure R between the particle and the plane.

If the first condition is satisfied, we have

$$R = W \cos \alpha - W \sin \alpha \,.\, \tan \theta$$

$$= W \frac{\cos (\alpha + \theta)}{\cos \theta} \quad . \quad . \quad . \quad . \quad \text{(ii)}$$

The first condition shows that $\cos\theta$ must be positive, and hence the second condition shows that $\cos(\alpha+\theta)$ must also be positive. These two conditions are possible only when

$$\theta \text{ lies between } -\frac{\pi}{2} \text{ and } \left(\frac{\pi}{2}-\alpha\right),$$

that is, as before, when OL lies between OM' and ON'.

For any given value of θ between these limits, the equations (i) and (ii) give the corresponding values of P and R.

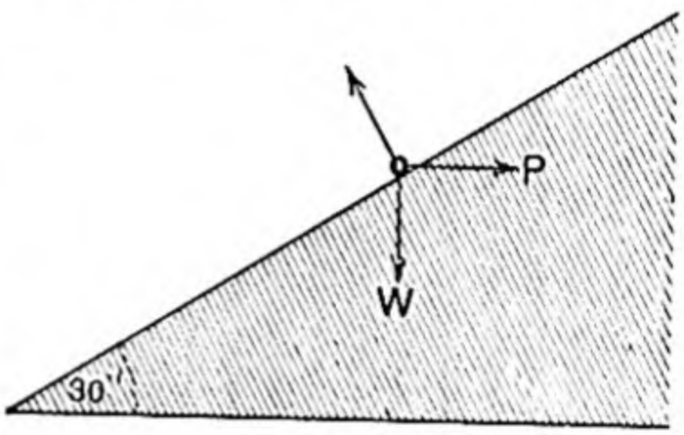


FIG. 60.

Example.—Suppose it is required to determine what *horizontal* force will support a body of weight W on a smooth plane inclined at an angle of 30° to the horizontal.

Here, resolving along the plane, we have

$$P\cos 30^\circ = W\sin 30^\circ.$$

$$\therefore\ P = W\tan 30^\circ = \tfrac{1}{3}W\sqrt{3}.$$

70. Pressure between Two Smooth Bodies.—In the case of two smooth bodies resting in contact, the mutual pressure at every point of contact is *normal* to both surfaces at that point.

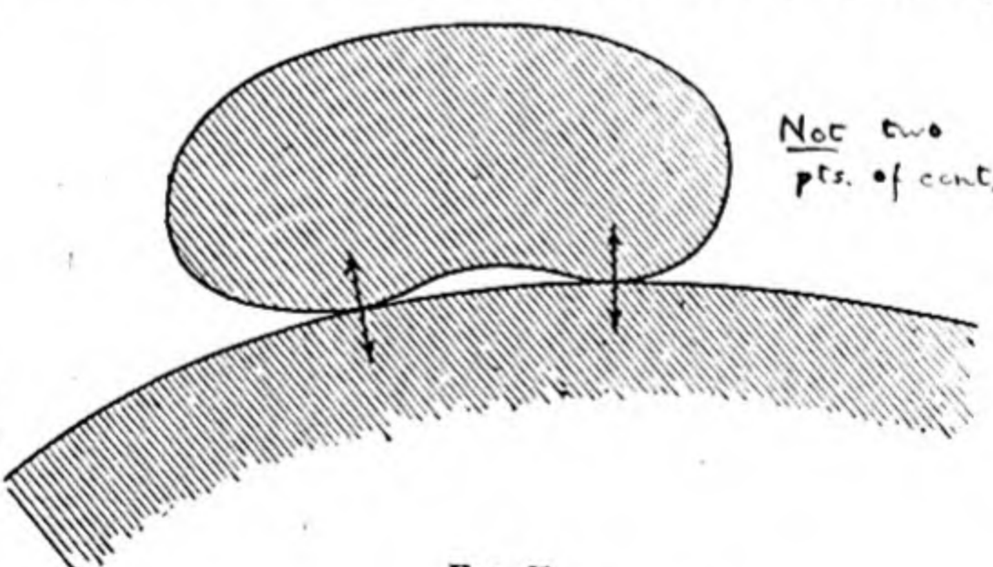


FIG. 61.

Example.—Suppose that a uniform spherical ball, of weight W and radius r, rests on a smooth inclined plane, being supported by a fine string of given length l, connecting a point B in the surface of the sphere with a point C in the surface of the plane.

In the position of equilibrium, let θ be the inclination of the string to the plane, and let the sphere rest against the plane at the point A.

The forces which act upon the sphere are:—Its resultant weight W, acting vertically downwards through its centre O; the reaction R of

the plane, acting along the normal AO; and the tension T of the string, acting along BC. The first two of these forces act along lines which intersect at O; hence the line of action of the third must pass

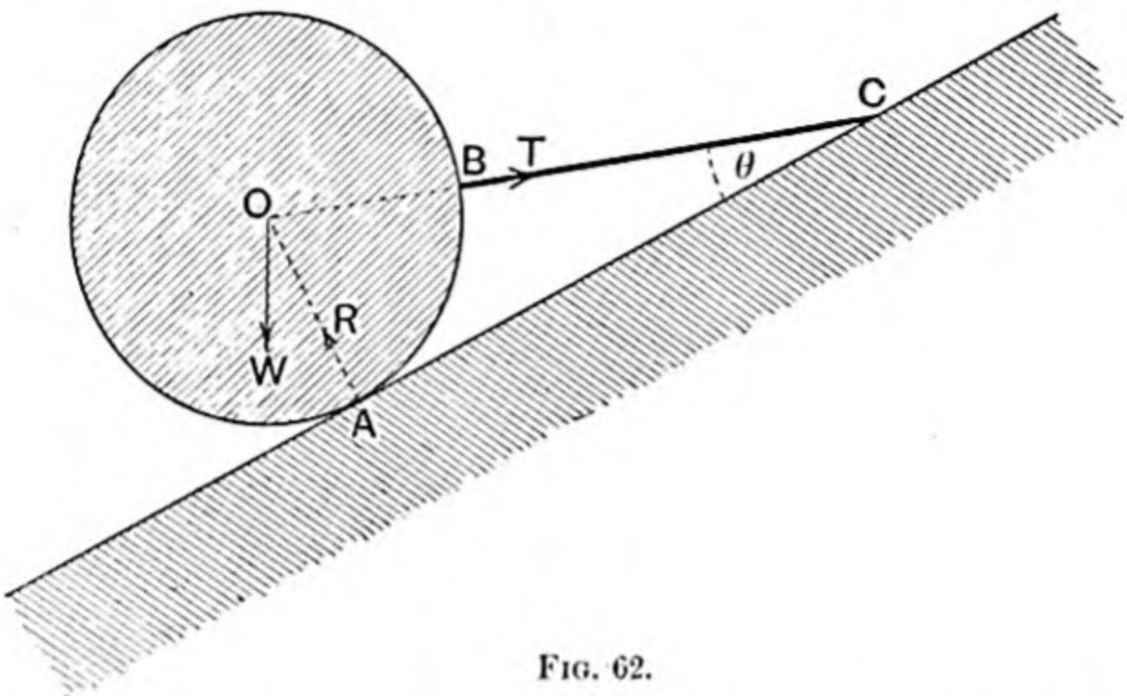


FIG. 62.

through O, so that OBC is a straight line. Thus we see that θ is given by the equation $\sin\theta = \frac{r}{r+l}$.

The forces T and R may now be determined, either by resolving along and perpendicular to the plane, or by Lamé's Theorem.

EXAMPLES IX

1. A small ring, of weight W, is capable of sliding freely on a fixed, smooth, vertical rod, and is supported by a fine light string, of length 20 inches, connecting it with a fixed point situated at a distance of one foot from the rod. Find the tension of the string and the pressure between the ring and the rod.

2. A is the lowest point of a smooth circular hoop fixed in a vertical plane. A small ring of weight W rests on the hoop at B, being supported by a horizontal force. Find the magnitude of this force and the pressure between the ring and the hoop, if the arc AB subtends an angle α at the centre.

3. A small ring, of weight W, is capable of sliding freely on a smooth circular hoop, of radius r, fixed in a vertical plane. It is supported by a fine light string, of length $a<2r$, attaching it to the highest point of the hoop. Find the tension of the string, and the pressure between the hoop and the ring.

4. A small ring, of weight W, which can slide freely upon a smooth thin rod AB, is attached to the end A of the rod by a fine light string. If the rod is held, with A uppermost, in a position inclined at an angle α to the vertical, find the tension of the string and the pressure between the rod and the ring.

5. A small ring C, of weight W, is capable of sliding freely on a smooth thin rod AB, fixed with A uppermost in a position inclined

at an angle α to the vertical. It is supported by a fine light string connecting it with a point D situated vertically above A, such that $AD = AC$. Find the tension of the string and the pressure between the rod and the ring.

6. A small ring, of weight W, is capable of sliding freely on a smooth circular hoop, of radius r, fixed in a vertical plane. It is supported by a fine light string, of length $a < r\sqrt{2}$, attaching it to one extremity of the horizontal diameter of the hoop. Find the tension of the string, and the pressure between the hoop and the ring.

7. A small ring of weight W, which can move without friction on a circular wire fixed in a vertical plane, is in equilibrium at a point P, on the lower half of the wire, under the action of a force R in the direction of the tangent at P to the wire. If the pressure of the ring on the wire is equal to $\frac{1}{2}W$, find the magnitude and direction of the force R.

8. Find the force necessary to sustain a particle of weight W placed on a smooth plane inclined at an angle α to the horizontal, (i) when the force is horizontal, and (ii) when it acts along the inclined plane.

In what direction must the force be applied, in order that there may be no pressure on the plane?

9. If a heavy body is supported on a smooth inclined plane by a force acting up the plane, show that the sustaining force, the resistance of the plane, and the weight of the body are proportional to the height, the base, and the length of the plane respectively.

10. If a heavy body is supported on a smooth inclined plane by a horizontal force, show that the sustaining force, the resistance of the plane, and the weight of the body are proportional to the height, the length, and the base of the plane respectively.

11. A body whose mass is 520 pounds is placed on a smooth inclined plane, the tangent of whose inclination to the horizontal is $\frac{5}{12}$; find the force necessary to sustain the body, (i) when this force is horizontal, and (ii) when it acts along the inclined plane.

12. A body of weight W is placed on a smooth plane inclined to the horizontal at an angle of two-thirds of a right angle, and is supported by a force whose line of action makes with the horizontal an angle half that of the plane. Find the magnitude of the force, and the pressure on the plane.

13. The *smallest* force which can support a body of weight W on a smooth inclined plane, is of magnitude P. Show that the *horizontal* force necessary to support the same body on the same plane is of magnitude $\dfrac{PW}{\sqrt{W^2 - P^2}}$.

14. Find what force, acting horizontally, will support a mass of 30 pounds resting on a smooth inclined plane, the base of the plane being three times its height; find also the pressure on the plane.

15. Find the greatest vertical height through which a force of 150 pounds' weight can raise a mass of 4 cwt. by drawing it up a smooth, sloping plank 20 feet in length.

16. What horizontal force will support a body of weight W on a smooth plane inclined at an angle of 45° to the horizontal?

If the inclination of the plane is increased by an angle θ, show that the sustaining force must be increased by a force $2W\left(\frac{\tan\theta}{1-\tan\theta}\right)$.

17. Show that the weight of the greatest mass, which a given force can sustain on a smooth inclined plane of given height, is proportional to the length of the plane.

18. A heavy body is supported on an inclined plane of height h by a force P acting up the plane. If the height is altered to h', the length of the plane remaining the same, what force acting up the plane is now necessary to support the same body?

19. A body of weight W rests on a smooth plane inclined at an angle α to the horizontal, and is supported by a force acting at an angle with the plane. If the resistance of the plane is equal to the sustaining force, show that the latter must be inclined at an angle 2α with the normal to the plane, and that its magnitude is $\frac{1}{2}W\sec\alpha$.

20. A particle of weight W is supported on a smooth plane, inclined at an angle α to the horizontal, by means of a string of length l attached to the particle and to the top of a peg of height h, standing out perpendicularly from the plane. Find the tension of the string.

21. A body of weight W is supported on a smooth plane, inclined at an angle α to the horizontal, by means of a force inclined at an angle α to the vertical; show that the pressure on the plane is equal to the force applied.

If the body is supported by a *horizontal* force, show that the pressure on the plane is twice as great as before.

22. A body of weight W is supported on a smooth plane, inclined at an angle α to the horizontal, by a fine string attached to a point in the plane; find the tension of the string.

If the string is altered, so as to be inclined at an angle α to the vertical, the inclination of the plane remaining the same as before, and if the pressure on the plane is changed from R to R', show that

$$R = 2R'\cos^2\alpha.$$

23. A body of weight W is supported on a smooth plane, inclined at an angle α to the horizontal, by two equal forces, one acting horizontally and the other up the plane. Show that the pressure on the plane is equal to W, and that the magnitude of each of the equal forces is $W\tan\frac{\alpha}{2}$.

24. A body of weight W is supported on a smooth plane, inclined at an angle α to the *vertical*, by two forces, one acting up the plane and the other in a horizontal direction tending to draw the body down the plane. If the upward force is of magnitude W, show that the horizontal force is of magnitude $W\tan\frac{\alpha}{2}$, and is equal to the pressure on the plane.

25. A mass of 24 pounds is supported on a smooth inclined plane by means of two forces—one of 15 pounds' weight acting up the plane, and the other of 7 pounds' weight acting horizontally. Find the ratio of the height of the plane to its length, and determine the pressure on the plane.

26. A fine uniform rod AB, of weight W, capable of turning freely in a vertical plane about its extremity A, which is fixed, rests in a horizontal position with its extremity B in contact with a smooth plane inclined at an angle α to the horizontal. Find the actions at A and B.

27. A rectangular box, containing a uniform spherical ball of weight W, stands on a horizontal table, and is tilted about one of its lower edges through an angle of 30°. Find the pressures between the ball and the box.

28. A uniform spherical ball, of radius 8 inches and weighing 10 pounds, is supported in contact with a smooth vertical wall by a fine light string, 9 inches long, connecting a point in the surface of the ball with a point in the wall. Find the tension of the string.

29. A uniform spherical ball, of weight W, rests on a smooth inclined plane, being supported by a fine string, of length equal to the radius of the sphere, connecting a point in its surface with a point in the plane. If the tension of the string is equal to the weight of the ball, find the inclination of the plane to the horizontal, and determine the pressure between the ball and the plane.

30. A uniform spherical ball, of weight W and diameter 5 inches, rests upon a smooth inclined plane, whose height is three-fifths of its length, being supported by a fine string, 4 inches long, connecting a point in the surface of the ball with a point in the plane. Find the tension of the string and the pressure between the ball and the plane.

31. A uniform spherical ball, of weight W and diameter d, is placed upon a smooth inclined plane, whose height is three-fifths of its length, and rests against a smooth horizontal rail fixed parallel to the plane. If the pressure upon the rail is $\frac{5}{8}W$, determine its perpendicular distance from the inclined plane, and find the pressure between the ball and the plane.

$\sin\theta = \frac{3}{5}\cdot\frac{1}{5/8} = \frac{24}{25} \therefore \cos\theta = \pm\frac{7}{25}$

$x = \frac{d}{2}\left(1 \pm \frac{7}{25}\right)$ $\quad R = W\left(\frac{5}{8} \pm \frac{3}{8}\cdot\frac{7}{25}\right)$

71. Fine Light String on a Smooth Surface.—Suppose that a fine light string, in a state of tension, rests with a portion of its length in contact with a smooth surface. The surface, being smooth, presses the string at every point of contact in the direction of the normal at that point.

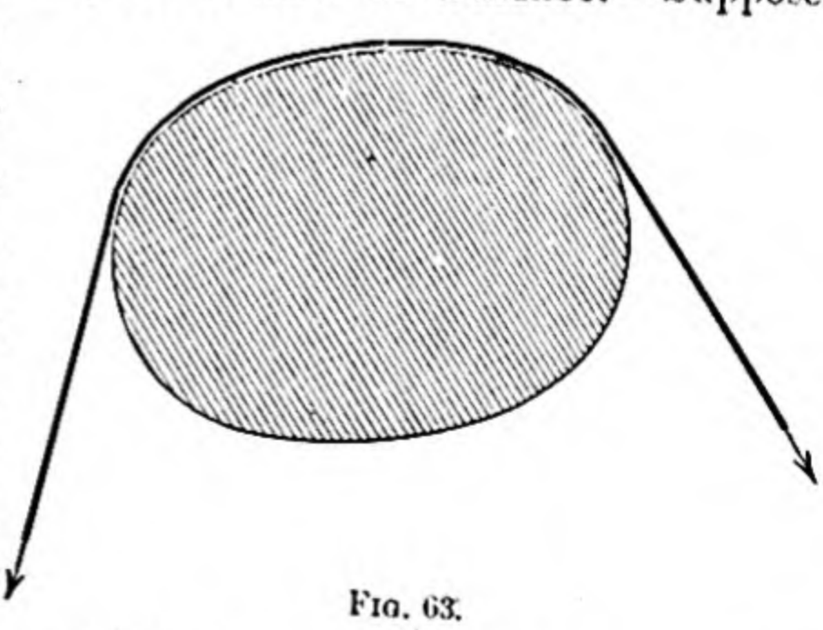


FIG. 63.

It is usual for the beginner to *assume* that under these circumstances the tension of the string is the same at every point. For a proof see the Author's *Elementary Geometrical Statics*.

72. Pressure between String and Peg.—*To find the resultant pressure between a fine light string, in a state of tension, and a smooth peg round which it passes.*

Let the fine light string BAC pass round a smooth peg at A, and let its tension at every point be of magnitude T.

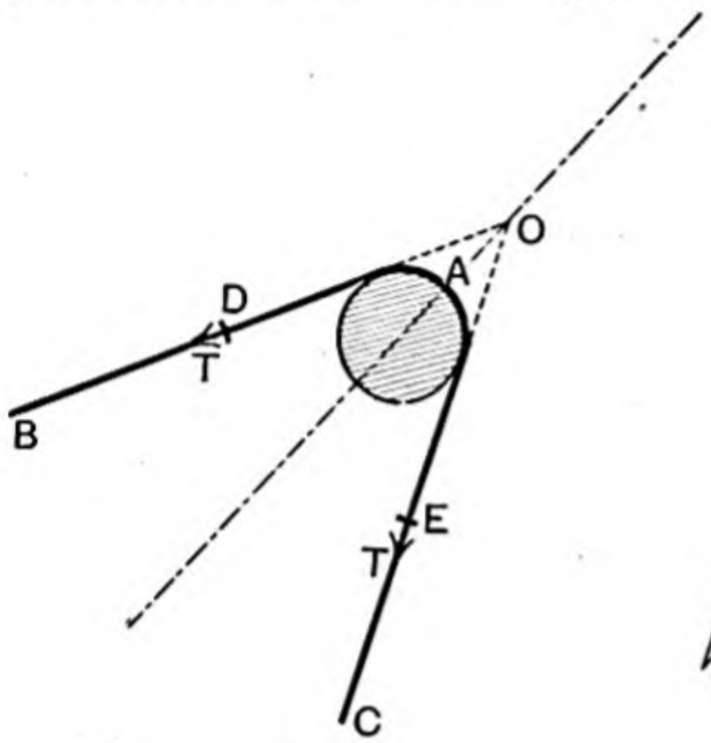


FIG. 64.

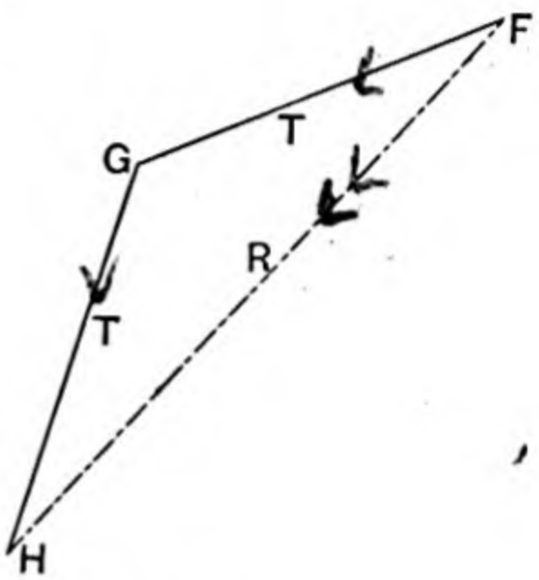


FIG. 64a.

Consider the equilibrium of the portion of string DAE in the neighbourhood of the peg. The forces which act externally upon this portion are :—

- At D, a tension T along DB.
- At E, a tension T along EC.
- At every point of contact with the peg, a pressure in the direction of the normal at that point.

Let FG and GH be drawn, each of length T units, in the directions of DB and EC respectively. Then the first two forces are equivalent to a force R represented by FH, acting along a line through O, the point of intersection of DB and EC. The resultant pressure of the peg upon the string balances this force R, and is therefore represented by HF, and acts through O. Also the resultant pressure of the string upon the peg is represented by FH and acts through O.

Analytically.—Let a be the angle between the two straight portions of the string. Then the angles F and H are each $\frac{1}{2}a$, and FH is of length $2T\cos\frac{1}{2}a$. Thus the resultant pressure between the string and the peg is of magnitude $2T\cos\frac{1}{2}a$, and acts along the bisector of the angle between the straight portions of the string.

Example.—The fine light string ABC, with one end fixed at A, rests against a (small) smooth peg at B, and supports at C a mass of weight W. The resultant pressure between the string and the peg acts along the bisector of the angle ABC and is of magnitude

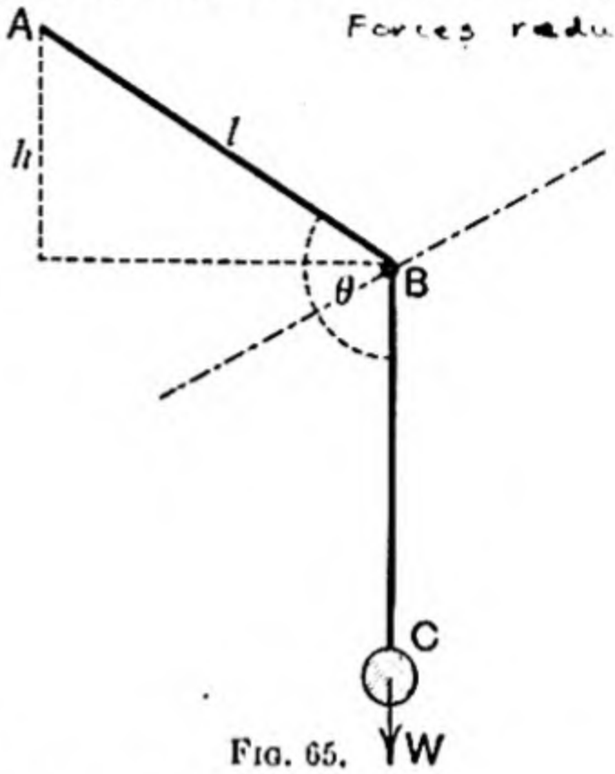


Fig. 65.

$$2W\cos\frac{\theta}{2}$$

$$=2W\sqrt{\frac{1+\cos\theta}{2}}$$

$$=2W\sqrt{\frac{1-\frac{h}{l}}{2}}$$

$$=W\sqrt{\frac{2(l-h)}{l}}.$$

73. Ex. 14.—*A fine light string $ABCD$ is attached at one extremity to a fixed point A. It passes through a small smooth ring B, of weight w, and over a smooth peg C, and supports at its extremity D a mass of weight W. It is required to determine the pressure between the string and the peg in the position of equilibrium.*

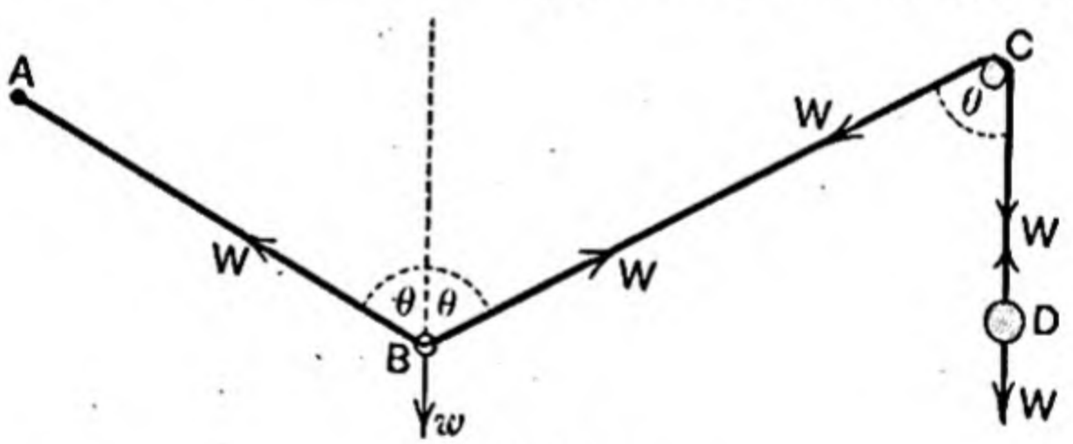


Fig. 66.

In the position of equilibrium, D hangs vertically below C, and the tension of the string is W throughout.

Considering the equilibrium of the portion of matter in the neighbourhood of B, we see that the strings AB and BC are inclined to the vertical at the same angle θ, given by the equation

$$2W\cos\theta = w.$$

This determines the position of equilibrium.

The pressure between the string and the peg is along the bisector of the angle BCD, and is of magnitude

$$2W\cos\frac{\theta}{2}$$

$$= 2W\sqrt{\frac{1+\cos\theta}{2}}$$

$$= 2W\sqrt{\frac{2W+w}{4W}}$$

$$= \sqrt{(2W+w)\,W}.$$

74. Ex. 15.—*A fine light string ACB passes through a small smooth ring C, of weight W, and has its extremities attached to two given points A and B. It is required to find the magnitude of the horizontal force which, applied to the ring, will cause it to rest in a given position in the vertical plane containing A and B, and to determine the tension of the string.*

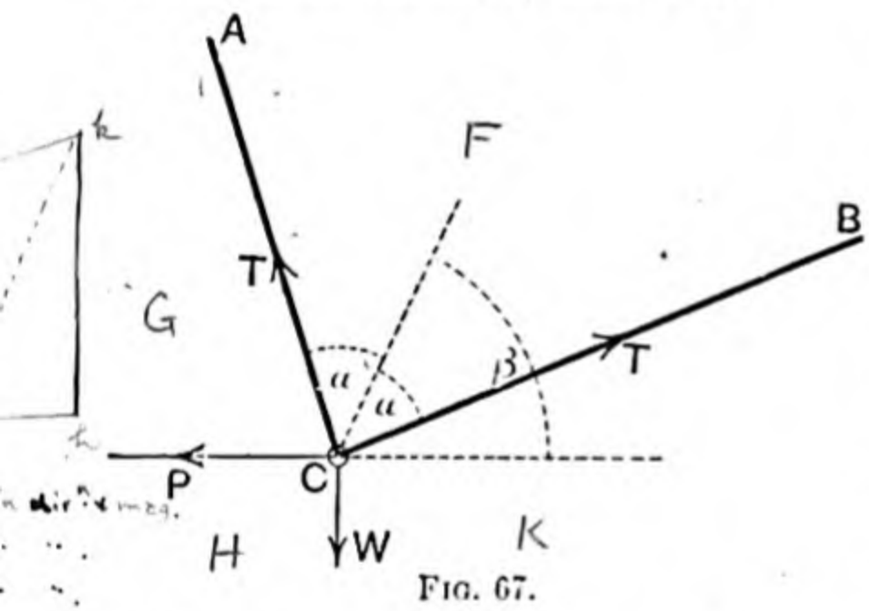


FIG. 67.

The tension of the string is the same throughout. Let its measure be T, and let P be the measure of the horizontal force applied to the ring.

Let the known angle ACB be of magnitude 2α, and let its bisector make an angle β with the horizontal.

The forces acting externally upon the portion of matter in the neighbourhood of C are—the force W acting vertically downwards, the horizontal force P, and the equal forces T acting along CA and CB. The last two forces can be replaced by a single force of magnitude $2T\cos\alpha$, acting along the bisector of the angle ACB.

Hence, resolving vertically and horizontally, we have

$$\begin{cases} 2T\cos\alpha\sin\beta = W \\ 2T\cos\alpha\cos\beta = P \end{cases}$$

$\therefore$ eliminating T,

$$P = W\cot\beta$$

Also,

$$T = \tfrac{1}{2}W\sec\alpha\operatorname{cosec}\beta.$$

[For a solution by geometrical construction see Art. 52 of the Author's *Elementary Geometrical Statics.*]

EXAMPLES X

1. One end B of a fine light cord is fixed; the cord passes over a small smooth fixed peg A in the horizontal line through B, and supports at its other end C a mass of weight P; find the magnitude and direction of the pressure on the peg.

2. A fine light string has one end attached to a fixed point A. It passes over a small smooth peg B, distant 18 inches from A and 7 inches farther from the ground than A, and supports at its other extremity a mass of 12 pounds. Find the pressure between the string and the peg.

3. A fine light string, of length 50 inches, is passed through a small smooth ring of inappreciable weight, to which is attached a mass weighing 24 pounds. The extremities of the string are attached to two fixed points, situated 40 inches apart in a horizontal line. Prove that, in the position of equilibrium, the ring rests at the middle point of the string, and find the tension of the string.

4. A fine light string has its extremities attached to two masses, each weighing 5 pounds, and passes over two small smooth pegs A and B. The peg B is situated 7 inches farther from the ground than A, and 24 inches horizontally to the right of A. Find the resultant pressures of the string upon the pegs.

5. A fine light string has one extremity attached to a fixed point O, passes over a small smooth peg A, and supports at its other extremity a mass of weight W. The peg A is situated 5 inches farther from the ground than O, and one foot horizontally to the right of O. Find the pressure between the string and the peg.

6. A fine light string has its extremities attached to two masses, each of weight W, and passes through two small smooth rings A and B, fixed in space at a distance l apart, the ring B being situated at a height h farther from the ground than the ring A. Find the pressures between the strings and the rings.

7. A mass of weight W is attached to an endless fine light string, of length $3a$, which is supported on two small smooth pegs situated at a distance a apart in a horizontal line. Find the pressure between the string and each peg.

8. A fine light string, of length l, passes through a small smooth ring of weight W, and has its extremities fixed at two points situated at a distance c apart in a horizontal line. Find the tension of the string in the position of equilibrium.

9. A fine light string, 31 inches long, passes through a small smooth ring of mass 4 ounces, and has its extremities fixed at two points situated 25 inches apart in a horizontal line. Find the tension of the string in the position of equilibrium.

10. A mass of weight W is attached to an endless fine light string, of length l, which is supported on two small smooth pegs situated at a distance c apart in a horizontal line. Show that the pressure between the string and each peg is of magnitude

$$W\sqrt{\frac{l-c}{2(l-2c)}}.$$

11. A fine light string, 31 inches long, passes through a small smooth ring of mass 4 ounces, and has its extremities fixed at two points situated 25 inches apart in the same horizontal line. Find the magnitude of the horizontal force which, applied to the ring, will cause it to rest at a point 7 inches from the nearer end of the string. Also determine the tension of the string.

12. A fine light string, of length 5 feet, passes through a small smooth ring, to which a mass of weight W is attached, and has its extremities fixed at two points situated 4 feet apart in a horizontal line. Find what horizontal force applied to the ring will cause it to rest at a point 2 feet from one end of the string. Also determine the tension of the string.

13. A fine light string, of length 7 inches, has its extremities attached to two points A and B, situated 5 inches apart on a smooth horizontal table. To a point C of the string, 4 inches from A, is knotted another fine light string CD, which passes over the smooth edge of the table and supports at its free extremity a mass of weight W. Find the tensions in AC and BC, supposing that DC produced passes through the middle point of AB.

14. A fine light string, of length $a+b$, has its extremities attached to two points A and B, situated at a distance c apart on a smooth horizontal table. To a point C of the string at a distance a from the extremity B, is knotted another fine light string CD, which passes over the smooth edge of the table and supports at its free extremity a mass of weight W. Find the tensions in AC and BC, supposing that DC produced passes through the middle point of AB.

15. A fine light string $ABCD$ has its upper extremity A fixed, and passes round two small smooth pegs at B and C, supporting at its free end D a mass of weight W. If C is vertically below A, CA and CB each of length a, and AB of length c, determine the pressures between the string and the pegs.

16. A fine light string, of length l, is passed through a small smooth ring of no appreciable weight, and is attached at its extremities to two fixed points A and B at a distance c apart. A given force P is applied to the ring in a direction inclined at a given angle α with BA. Show that, in the position of equilibrium, the two parts of the string are inclined to one another at an angle $2\sin^{-1}\left(\frac{c}{l}\sin\alpha\right)$. Show also that the tension of the string is of magnitude

$$\frac{l\sin\alpha}{2\sqrt{l^2\sin^2\alpha - c^2}}\cdot P.$$

17. A fine light string, of length l, is passed through a small smooth ring C of no appreciable weight, and is attached at its extremities to two fixed points A and B. The point B is situated at a distance k farther from the ground than A and at a distance h horizontally to the right of A. If a mass of weight W is suspended from C, show that, in the position of equilibrium, C rests at a distance $\frac{1}{2}\sqrt{l^2-h^2}$ below, and at a distance $\frac{1}{2}\frac{hk}{\sqrt{l^2-h^2}}$ to the left of the middle point of

AB. Prove also that the tension of the string is of magnitude

$$\tfrac{1}{2}\frac{l}{\sqrt{l^2-h^2}}W.$$

18. A, B, C are three smooth pegs fixed in a vertical plane, A being vertically above the middle point of BC, which is horizontal. A fine endless string passes round the three pegs and supports at a point D below BC a mass of weight W. If each portion of string is inclined at an angle α to the vertical, determine the tension of the string and the pressures on the pegs.

19. A, B, C are three smooth pegs fixed in a vertical plane, A being 3 feet vertically below B and 4 feet horizontally to the right of C. An endless fine light string, 13 feet long, passes round the three pegs and supports at a point below CA a mass of weight W. Find the tension of the string and the resultant pressures upon the pegs.

20. An endless fine light string $CABC$, of length $a+b+c$, passes round two small smooth pegs at A and B, situated at a distance c apart on a smooth horizontal table. In what direction must a horizontal force P be applied to the point C, situated at a distance b along the string from A, in order that the string may remain stretched without slipping over the pegs? If the force is applied in this direction, determine the pressures upon the pegs.

21. A fine light string, placed across two smooth parallel bars situated at a distance a apart in a horizontal plane, supports at each extremity a mass of weight W; and another mass of weight w is attached to a point A of the string between the bars. Find the depth of A below the level of the bars in the position of equilibrium, and determine the magnitude of the pressure on each bar.

22. A fine light string $ABCD$, fixed at one extremity A, passes through a small smooth ring B, and over a small smooth peg C, fixed at a distance of 2 feet in a horizontal line from A, and supports at its extremity D a mass of 5 pounds. In the position of equilibrium the ring rests at a distance of $3\frac{1}{2}$ inches below AC. Determine the mass of the ring and the pressure upon the peg.

23. A, B, C are three points in a vertical plane, A and C lying on opposite sides of the vertical line through the highest point B. A fine light string $ADBCDE$, having one end fixed at A, passes in succession through a light smooth ring D, round pegs at B and C, again through the ring, and is attached to a heavy mass at its free extremity E. Prove that, in the position of equilibrium, the ring and the mass hang vertically below B.

24. A fine light string, fixed at one extremity O, passes over a small smooth peg A, situated at a given distance a in a horizontal line from O, and supports at its free extremity a mass of known weight W. Another mass of known weight w is attached to the string at some point X between O and A. Determine the length of OX, in order that it may rest inclined at a given angle α with OA. Show that there may be two solutions, but that in both cases AX is of length $a\left(\frac{W}{w}\right)\tan\alpha$.

*75. Ex. 16.—*A fine straight smooth rod AB, on which a small smooth ring P, of no appreciable weight, is capable of sliding, is fixed at an inclination α to the vertical with the end A uppermost. A fine light string CPD has one end fixed at the point C, situated at a given distance b vertically below A, and supports at its other extremity D a mass of given weight W. It is required to determine the tension of PC, and the pressure between the rod and the ring in the position of equilibrium,* (i) *when the string is attached to the ring P, so that PC is of given length l, and* (ii) *when the string is free to slip through the ring.*

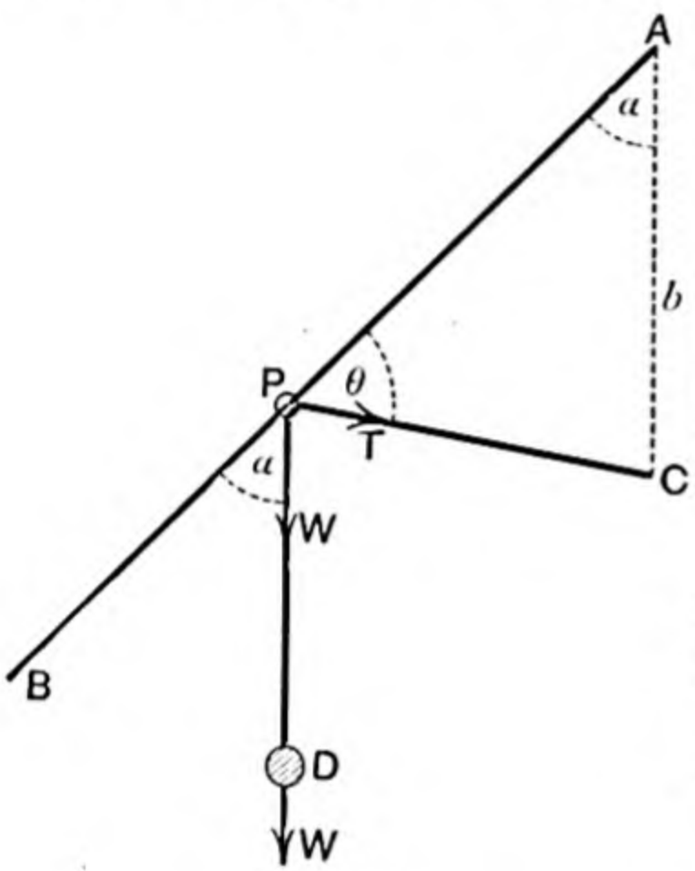


FIG. 68.

In the position of equilibrium, the two portions of the string are straight, and the part *PD* hangs vertically downwards, its tension being of magnitude *W*.

Let *T* be the tension of the string *PC*, and θ its inclination to the rod.

(i) The angle θ is determined from the equation

$$\frac{\sin\theta}{b} = \frac{\sin\alpha}{l}.$$

i.e. $$\theta = \sin^{-1}\left(\frac{b\sin\alpha}{l}\right) \text{ or its supplement.}$$

Resolving along the rod, we have

$$T\cos\theta = W\cos\alpha,$$

giving $$T = W\cos\alpha\sec\theta.$$

This result shows that the obtuse value for θ must be rejected, as it would require that *T* should be negative.

Also, resolving perpendicularly to the rod, we see that the pressure required is $T\sin\theta + W\sin\alpha$,

where *T* and θ have been already determined.

(ii) In this case the tension of the string is of magnitude *W* throughout, *i.e.* $T = W$. Resolving along the rod, we have

$$W\cos\theta = W\cos\alpha$$

$$\therefore \quad \theta = \alpha$$

$$\therefore \quad CP = CA = b.$$

Also, resolving perpendicularly to the rod, we see that the pressure required is

$$2W \sin \alpha.$$

✓*76. Ex. 17.—*A straight smooth rod AB, of no appreciable weight and of length a, is capable of turning freely about the lower end A, which is fixed, and carries a mass of weight W suspended from the other end B. A small smooth ring P, of no appreciable weight, is capable of sliding along the rod, and is attached to one extremity of a fine light string, which passes over a small smooth peg C, situated at a given distance b vertically above A, and supports at its other extremity a mass of known weight w. It is required to determine the position of equilibrium.*

The tension of the string which passes round the peg C is of magnitude w throughout. The only forces acting on the ring P are this tension w and the normal reaction of the rod. Hence, in the position of equilibrium, CP is perpendicular to AB.

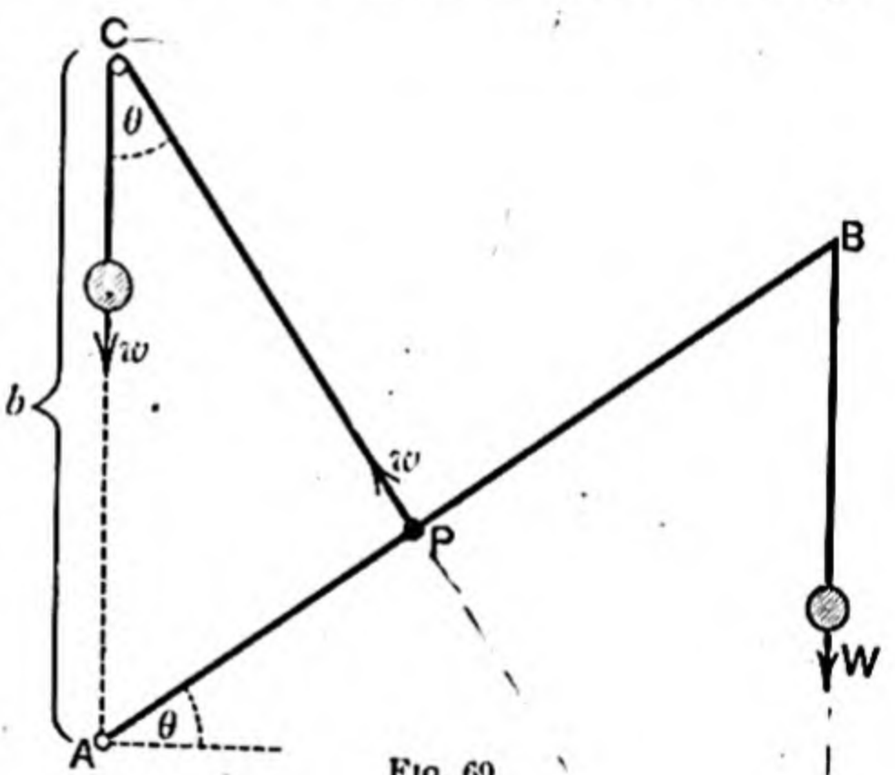


FIG. 69.

Let θ be the inclination of the rod to the horizontal. Then, also,

$$\widehat{ACP} = \theta.$$

Consider the equilibrium of the rod and the ring together as one system. The forces acting externally upon the system are, W acting vertically downwards at B, w acting along PC, and the constraint at A. The perpendiculars from A upon the lines of action of the first two of these three forces are of lengths $a \cos \theta$ and $b \sin \theta$ respectively.

Hence,
$$W.a \cos \theta = w.b \sin \theta$$

$$\therefore \tan \theta = \frac{Wa}{wb}.$$

This determines the angle θ. Also the distance of P from A is $b \sin \theta$, so that the position of equilibrium becomes fully known.

*77. Ex. 18.—*CADB is a thin lamina, whose centre of gravity is at G, the sides CA and CB being straight. It is supported in a vertical plane with CA and CB resting against two small smooth pegs H and K respectively, situated at a distance c apart in a horizontal line. It is required to determine the position of equilibrium.*

Let CG be of length a and make angles α and β with CA and CB respectively. Let θ be the inclination of CG to the horizontal in the position of equilibrium.

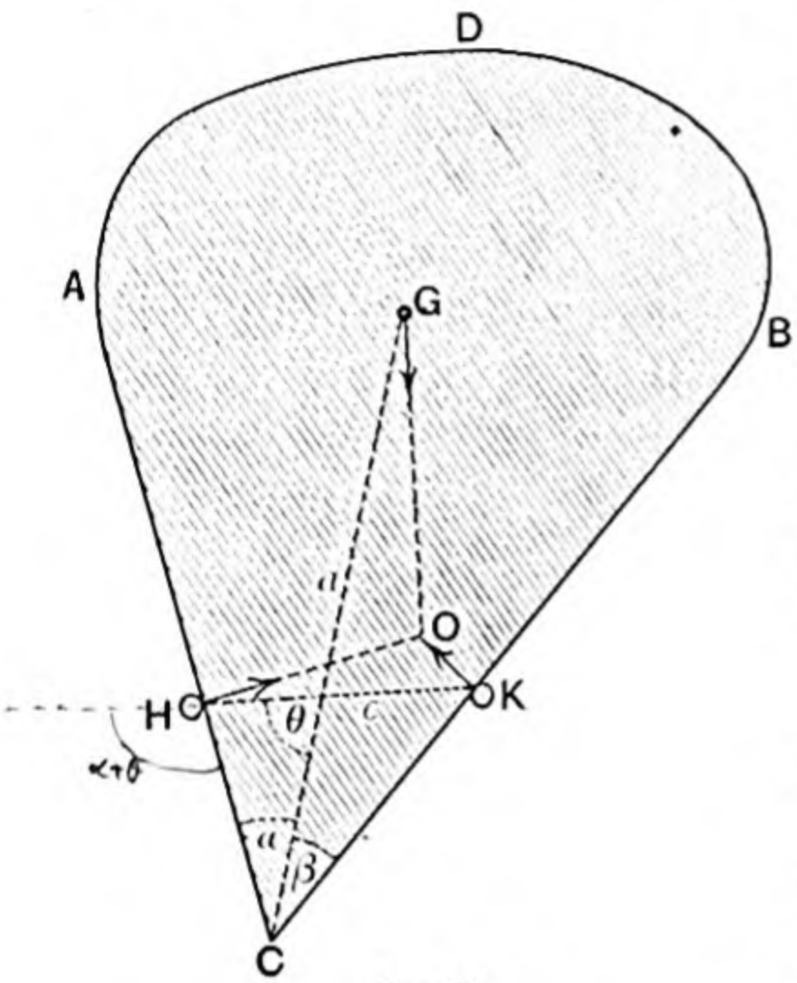


FIG. 70.

Draw HO and KO perpendicular to CA and CB respectively, to meet in O. Then the reactions of the pegs are along HO and KO, and for equilibrium it is necessary and sufficient that G and O should be in the same vertical line.

Now the sum of the projections of CH, HO, OG in any direction is equal to the projection of CG in that direction, each projection having its own proper sign. Projecting horizontally, this gives

$$CH\cos(\theta+\alpha)+HO\sin(\theta+\alpha)=a\cos\theta.$$

Now
$$\frac{CH}{HK}=\frac{\sin CKH}{\sin HCK}=\frac{\sin(\theta-\beta)}{\sin(\alpha+\beta)}.$$

Also
$$\frac{HO}{HK}=\frac{\sin HKO}{\sin HOK}=\frac{\cos(\theta-\beta)}{\sin(\alpha+\beta)}.$$

So that
$$CH=\frac{c\sin(\theta-\beta)}{\sin(\alpha+\beta)}, \quad \text{and } HO=\frac{c\cos(\theta-\beta)}{\sin(\alpha+\beta)}.$$

Thus, we have

$$c\sin(\theta-\beta)\cos(\theta+\alpha)+c\cos(\theta-\beta)\sin(\theta+\alpha)=a\cos\theta\sin(\alpha+\beta).$$

i.e.
$$c\sin(2\theta+\alpha-\beta)=a\cos\theta\sin(\alpha+\beta).$$

This is an equation to determine the angle θ.

*EXAMPLES XI

1. A small ring C rests upon a fixed smooth horizontal rod AB, whose length is 13 feet. To the ring are attached two strings, one of which is 7 feet long and has its other extremity fixed at A, while the other passes over a smooth hook, situated 8 feet below B, and supports a mass of 20 pounds. Find the tension of the string AC.

2. A small ring C rests upon a fixed smooth horizontal rod AB, whose length is 7 feet. To the ring are attached two strings, one of which is 5 feet long and has its other extremity fixed at D, situated 4 feet vertically below A; the other passes over a small smooth hook, situated 3 feet vertically below B, and supports a mass of 30 pounds. Find the tension of the string CD.

3. A smooth straight rod AB is fixed in a position inclined at an angle α to the vertical with B uppermost. A small smooth ring C of weight w, capable of sliding freely along the rod, is connected with B by a fine light string; a second string is attached to the ring, and passes over a small smooth peg D fixed vertically below A, supporting at its free end a mass of weight W. If $AC=AD$, determine the tension of BC and the pressure between the ring and the rod.

4. A smooth straight rod AB is fixed in a position inclined at an angle α to the vertical with A uppermost. A fine light string of length l, one end of which is fixed at A, passes through a small smooth ring D, and is attached at its other extremity to another small smooth ring C capable of sliding freely along the rod. If a mass of weight W is suspended from D, determine the magnitude of the angle ADC and the tension of the string in the position of equilibrium, supposing that the rings C and D are of no appreciable weight. Find also the distance of C below A.

5. A small ring, of weight w, is capable of sliding freely on a smooth circular hoop, of radius r, fixed in a vertical plane. It is supported by a fine light string, of length $a<2r$, attaching it to the highest point of the hoop. A second string is attached to the ring and passes over a small smooth peg situated at the lowest point of the hoop, supporting at its other extremity a mass of weight W. Find the tension of the first string.

6. AB is the horizontal diameter of a smooth circular hoop, of radius r, fixed in a vertical plane. A small ring, of weight w, capable of sliding freely on the upper part of the hoop, is connected with A by a fine light string of length $a<2r$. A second string is attached to the ring and passes over a small smooth peg situated at B, supporting at its other extremity a mass of weight W. Find the tension of the first string.

7. A uniform straight rod ACB, of weight W and length $4a$, can turn freely about a hinge C, and rests with its lower end B in contact with a smooth inclined plane, the line of greatest slope through B being in the same vertical plane as the rod. If AC is of length a, and if the rod and the plane are each inclined at an angle of 30° to the horizontal, determine the pressure at B and the reaction of the hinge.

8. A uniform rod AB, of weight W and length $2a$, rests with its

lower end A in contact with a smooth vertical wall, being inclined to the wall at an angle α. It is supported by a fine light string connecting the point C of the rod with a point D in the wall, situated vertically above A. Find the length of the string and its tension, AC being of length b.

9. A uniform straight rod OA, of weight W, can turn freely about its lower end O, which is fixed, and rests with its extremity A against a smooth vertical wall. If O is at a distance $2a$ from the wall, and A at a height h above O, determine the pressure on the wall and the reaction of the hinge.

10. A uniform rod, of weight W, is supported by a fine string fastened to its ends, of double its own length, which passes over a smooth horizontal rail. Find the tension of the string, first, when the rod is hanging at rest in a vertical position, and secondly, when the rod is at rest in a horizontal position.

11. A triangular lamina ABC, of no appreciable weight, rests in a vertical plane with the middle points of the sides AB and AC in contact with two smooth pegs, the line joining them being horizontal. Determine the point in BC where a mass of weight W may be placed without disturbing the equilibrium; also, if AB, AC, BC are of lengths 4, 5, 6 feet respectively, find the pressure on the pegs in terms of W.

12. ABC is a rigid framework, of no appreciable weight, in the form of an equilateral triangle. It is supported in a vertical plane with CA and CB resting against two small smooth pegs H and K respectively, and with a mass of weight W suspended from D, the middle point of CB. If H is the middle point of AC, which is vertically downwards, determine the position of K in BC, and find the pressures on the pegs H and K.

13. A uniform beam AB, of weight W and length $2a$, is capable of turning freely about a fixed point A. To the lower end B is attached a fine light string, which passes over a small smooth peg C, situated at a distance b in a horizontal line from A, and supports at its free extremity a mass of weight X. Find the value of X in order that, in the position of equilibrium, the beam and the string may be equally inclined to the horizontal.

14. A smooth rod BC is passed through a small ring and placed upon a horizontal plane, with its ends attached to a fixed point A in the plane by two fine strings, AB and AC, which are tight. A horizontal force being applied to the ring, find its direction, and also the position of the ring on the rod, in order that equilibrium may not be disturbed, the lengths of BC, CA, AB being 25, 20, and 15 inches respectively.

15. A fine smooth wire BPC, in the form of a semicircle of radius 5 inches, is placed upon a smooth horizontal table, with its extremities attached to a fixed point O on the table by two fine strings, OB and OC, which are tight. A horizontal force is applied to a small ring P which can slide freely along the wire; find its direction, and also the position of the ring on the wire, in order that the whole may rest in equilibrium, the lengths of BO and OC being 8 and 6 inches respectively.

16. A straight smooth rod AB, of no appreciable weight and of length a, is capable of turning freely about the lower end A, which is

fixed, and carries a mass of weight W suspended from the other end B. A small smooth ring P, of no appreciable weight, is capable of sliding freely along the rod, and is connected with a point C, situated at a given distance b vertically below A, by means of a fine light string of length $l<b$. Find the tension of the string in the position of equilibrium.

17. A straight smooth rod AB, of no appreciable weight and of length 8 feet, is capable of turning freely about the lower end A, which is fixed, and carries a mass of 75 pounds suspended from the end B. A small smooth ring P, of no appreciable weight, is capable of sliding freely along the rod, and is attached to one extremity of a fine light string, which passes over a small smooth peg C, situated 5 feet vertically above A, and supports at its other extremity a mass of 35 pounds. Find the distance of P from A in the position of equilibrium, and determine the pressure between the string and the peg.

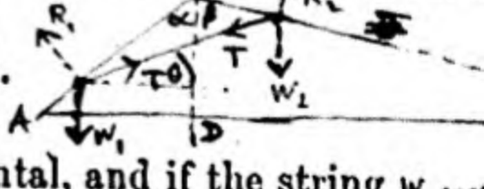
18. AB is a smooth straight wire fixed in a given inclined position with B uppermost. A small heavy ring of given weight, capable of sliding freely on the wire, is connected with B by a fine string of given length, which passes through a second small smooth ring of given weight, hanging freely on the string. Show how to determine the tension of the string, and the pressure between the ring and the wire.

19. AB is a smooth straight wire fixed in a position inclined at an angle α to the horizontal with B uppermost. A small heavy ring of weight W, capable of sliding freely on the wire, is connected with B by a fine string which passes through a second small smooth ring of equal weight, hanging freely on the string. Show that the tension of the string is $\frac{1}{2}W\sqrt{1+9\tan^2\alpha}$.

20. ABC is a smooth fixed wire, the portions AB and BC being straight and each inclined at an angle of 30° to the horizontal, and B situated at a higher level than A and C. P and Q are two heavy rings, each of weight W, which slide freely on the wire, P on AB and Q on BC. A fine light string connecting P and Q passes through a small smooth ring R, also of weight W. Determine the magnitude of the angle PRQ in the position of equilibrium, and find the tension of the string.

21. ACB is a smooth wire fixed in a vertical plane, the portions AC and BC being straight and inclined at angles α and β respectively to the vertical, and C being situated at a higher level than A and B. A small smooth ring of given weight W_1, capable of sliding freely on AC, is connected by a fine light string with a small smooth ring of given weight W_2, capable of sliding freely on BC. Show that, in the position of equilibrium, the string is inclined to the horizontal at an angle

$$\tan^{-1}\left(\frac{W_1\tan\beta - W_2\tan\alpha}{W_1+W_2}\right).$$



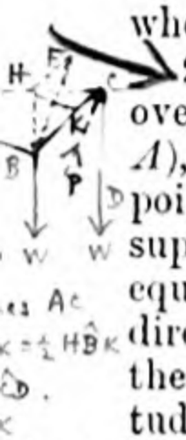
22. In the preceding example, if AB is horizontal, and if the string passes over a small smooth peg at C, show that equilibrium is possible only when

$$W_1 : W_2 = AC : BC.$$

23. Two small smooth rings, of weights W_1 and W_2, each capable of sliding freely on a smooth circular hoop fixed in a vertical plane, are connected by a fine light string. Show that, in the position of equilibrium in which the string is straight, it is inclined to the horizontal at an angle

$$\tan^{-1}\left(\frac{W_1 \sim W_2}{W_1 + W_2}\tan\frac{\alpha}{2}\right),$$

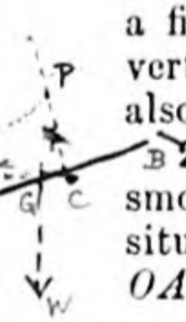
where α is the angle subtended by the string at the centre of the hoop.



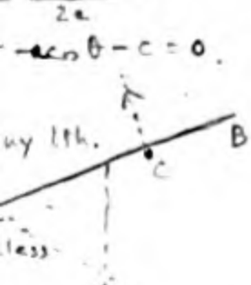
24. A fine light string $ABCD$ has one extremity A fixed, and passes over a small smooth peg at C (situated in the same horizontal line as A), supporting at its other extremity D a mass of weight W. To a point B of the string between A and C is knotted another string, supporting a mass of equal weight W. Prove that, in the position of equilibrium, (i) the angle DCB is double of the angle CAB; (ii) the direction of the pressure on the peg is at right angles to BA; (iii) if the pressure on the peg and the tension of the string BA are of magnitudes P and T respectively, then $P^2 + T^2 = 4W^2$.

25. AB is a smooth vertical rod on which a small heavy ring B of weight W is capable of sliding freely. The ring B is supported by a fine light string AOB, of length a, attached to the upper end A of the rod. To the middle point O of this string is attached a second fine string, which passes over a fixed smooth peg C and supports a mass of weight P. AC is horizontal and, in the position of equilibrium, the angle AOC is right. Prove that $AC = \frac{W}{P}a$, and find the tension of OB.

26. A uniform straight rod AB, of weight W, is capable of turning freely in a vertical plane about the extremity A which is fixed, and the other extremity B is attached to a mass of weight P by means of a fine light string which passes over a small smooth peg C, situated vertically above A. If $AC = AB$, find the position of equilibrium and also the action at the hinge.



27. A uniform rod ACB, of weight W and length $4a$, rests upon a smooth peg C, and its lower end A is attached to a fixed point O, situated in the same horizontal line with C, by means of a fine string OA. If OC and OA are each of length c, show that the inclination of the rod to the horizontal is $\cos^{-1}\dfrac{a + \sqrt{a^2 + 8c^2}}{4c}$, and that no position of equilibrium is possible unless a is greater than $\frac{1}{6}c\sqrt{6}$ and less than c.



CHAPTER VI

RESULTANT OF TWO PARALLEL FORCES—MOMENTS

78. Resultant of Two Parallel Forces. — *To find the resultant of two given forces acting along parallel lines upon the same rigid body.*

Let AB and BC represent the two parallel forces P and Q respectively, which act along the two parallel lines indicated in the diagram on the left.

Draw any straight line cutting the lines of action of the forces P and Q in M and N respectively. Draw BO parallel to MN, and, taking any point O in BO, draw AO and CO. Draw ML and NL parallel to AO and OC respectively, to meet in L.

We shall show that AC represents the resultant, and that L is a point in its line of action.

Case I.—Let the two forces P and Q be *like*, that is, parallel

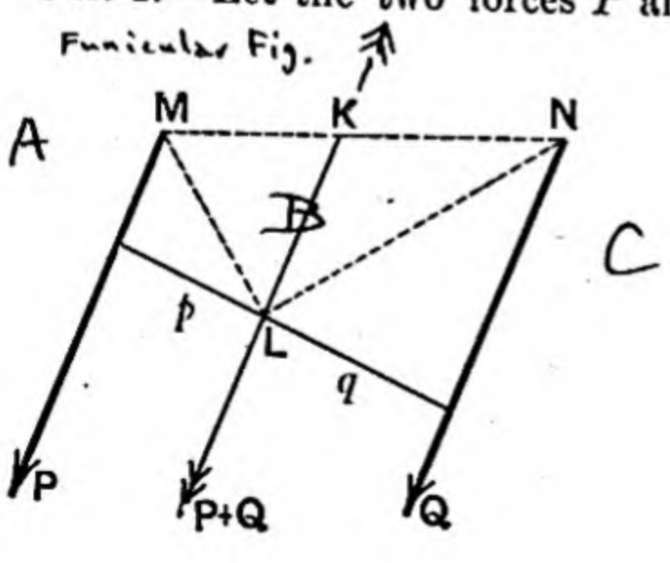


FIG. 71.

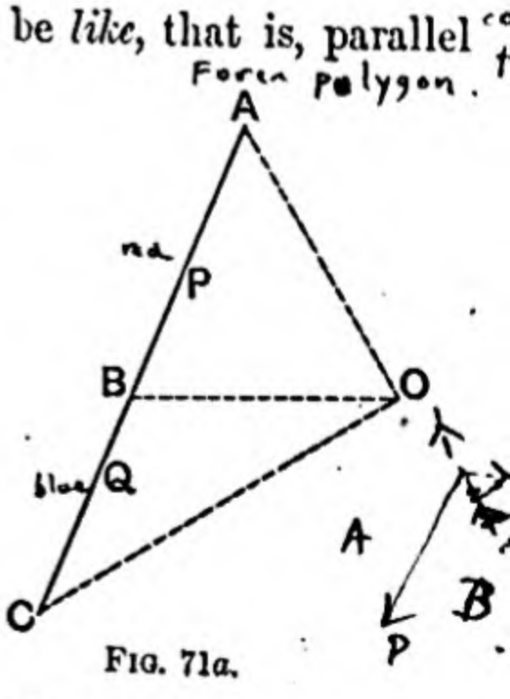


FIG. 71a.

and in the same direction. Then C will lie in AB produced.

The force P can be replaced by two forces represented by AO and OB, acting in the lines ML and NM respectively; and

the force Q can be replaced by two forces represented by BO and OC, acting in the lines MN and NL respectively. Let the forces P and Q be replaced by these pairs of components. Then the two forces represented by OB and BO acting in the line MN form a system in equilibrium and may be removed.

Hence the two given forces P and Q are equivalent to two forces represented by AO and OC, acting in the lines ML and NL respectively.

Hence AC represents the resultant and L is a point in its line of action.

Thus the resultant is of magnitude $P+Q$ and acts in the same direction as P and Q along a line through the point L.

Let the line of action of the resultant meet MN in K. Then we have

$$MK : KL = OB : BA$$
$$\text{and } KL : KN = BC : OB$$
$$\therefore \quad MK : KN = BC : BA = Q : P.$$

Thus the point K divides MN *internally* in the *inverse* ratio of P to Q.

Also, if the perpendiculars upon the lines of action of the forces P and Q from any point L are of lengths p and q respectively, then when L is in the line of action of the resultant, we have

$$p : q = MK : KN = Q : P$$
$$\therefore \quad Pp = Qq.$$

And, *conversely*, if $Pp = Qq$ and L lies *between* the lines of action of P and Q, then L is a point in the line of action of the resultant of P and Q.

Case II.—Let the two forces P and Q be *unlike*, that is, parallel and in opposite directions. Then, taking P as greater than Q, the point C will lie between A and B.

As before, the force P can be replaced by two forces represented by AO and OB, acting in the lines LM and NM respectively; and the force Q can be replaced by two forces represented by BO and OC, acting in the lines MN and NL respectively.

Hence, proceeding as before, we see that the two given forces P and Q are equivalent to two forces represented by AO and OC, acting in the lines LM and NL respectively.

Hence AC represents the resultant, and L is a point in its line of action.

Thus the resultant is of magnitude $P - Q$, and it acts in the same direction as the force P along a line through L.

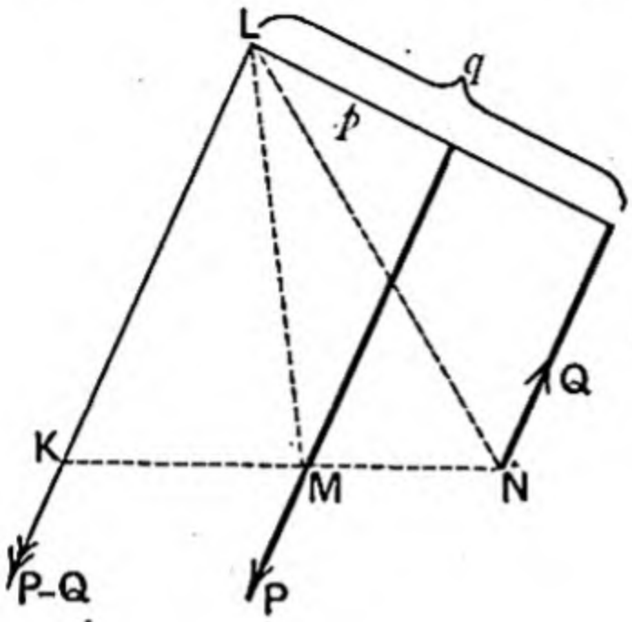


FIG. 72.

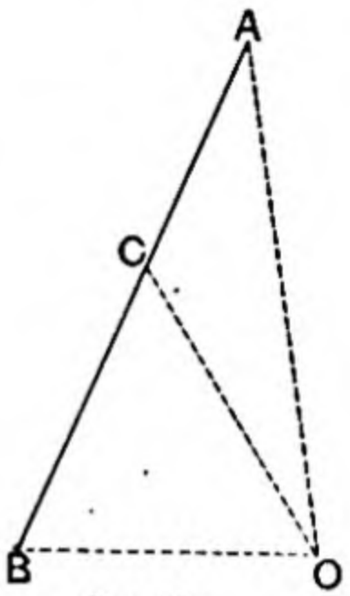


FIG. 72a.

Let the line of action of the resultant meet NM produced in K. Then we have

$$MK : KL = OB : BA$$
$$\text{and } KL : KN = BC : OB$$
$$\therefore \quad MK : KN = BC : BA = Q : P, \text{ a neg. ratio.}$$

Thus the point K divides MN *externally* in the *inverse* ratio of P to Q.

Also, if the perpendiculars upon the lines of action of the forces P and Q from any point L are of lengths p and q respectively, then when L is in the line of action of the resultant, we have

$$p : q = MK : KN = Q : P$$
$$\therefore \quad Pp = Qq.$$

And, *conversely*, if $Pp = Qq$ and L lies *not between* the lines of action of P and Q, then L is a point in the line of action of the resultant of P and Q. also P+Q are in opps. dirⁿ.

Example.—Suppose it is required to find the resultant of two parallel forces of 7 and 3 pounds' weight acting at points A and B respectively of a rigid body, where AB is of length 8 inches.

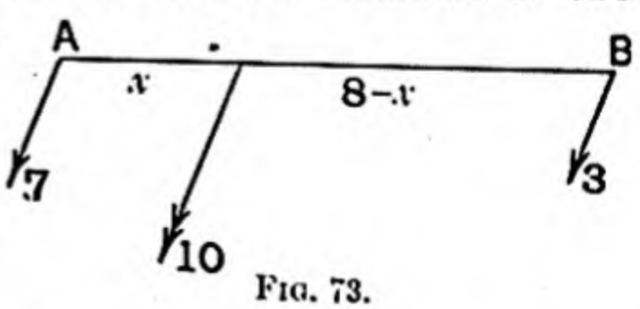


FIG. 73.

(i) If the given forces are *like*, the resultant is 10 pounds' weight, acting in the same direction as each of its components along a line which meets AB between A and B at a point distant x inches from A, where so that

$$7x = 3(8 - x),$$
$$x = 2{\cdot}4.$$

(ii) If the given forces are *unlike*, the resultant is 4 pounds' weight, acting in the same direction as the 7 pounds' weight along a

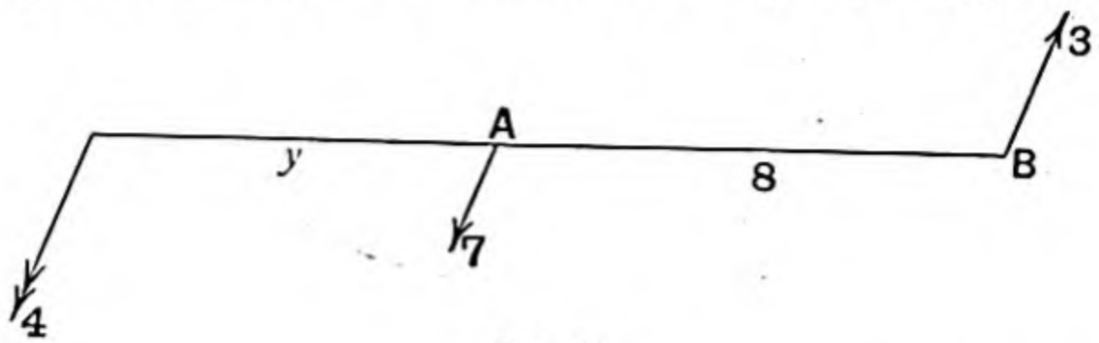


FIG. 74.

line which meets BA produced at a point distant y inches from A, where

$$7y = 3(8+y),$$

so that

$$y = 6.$$

79. Couple.—We see that the construction of the preceding article fails if P is equal and opposite to Q. Under these circumstances C coincides with A, and the straight lines drawn through M and N, parallel to OA and OC respectively, become parallel, so that there is no point L at a finite distance.

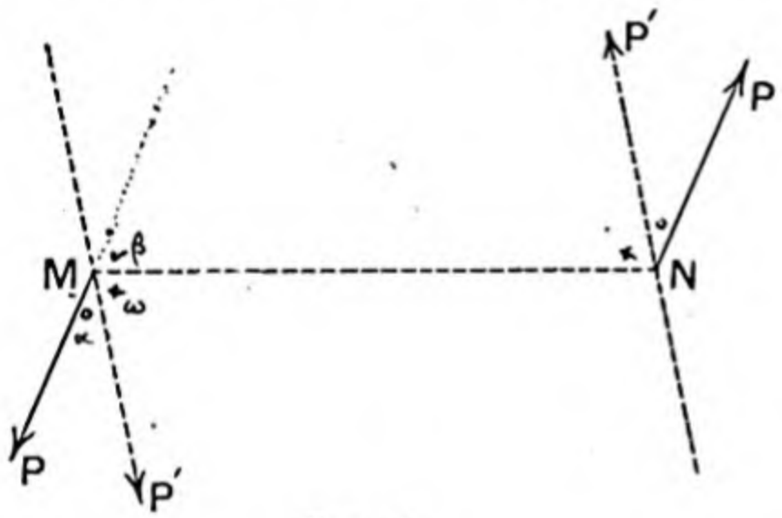


FIG. 75.

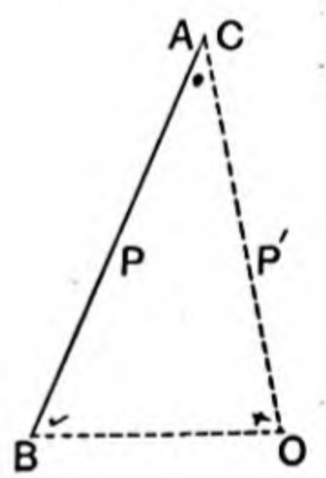


FIG. 75a.

It is, however, still true that the two given forces are equivalent to two forces represented by AO and OC, acting along lines through M and N respectively.

Thus, if AO contains P' units of length, we see that two equal and opposite forces P, represented by AB and BA, and acting at M and N respectively, are equivalent to two equal and opposite forces P', represented by AO and OA, and acting at M and N respectively, where BO is parallel to MN.

Def.—Two equal and parallel forces, acting in opposite directions along different lines, are said to form a *couple*.

80. *A couple cannot be in equilibrium with a single force.*—If the single force acts along a line parallel to the lines of action

of the two forces which constitute the couple, then, by the preceding article, the given couple can be replaced by another couple consisting of two equal and opposite forces acting along lines which are not parallel to the line of action of the single force.

Now, if three forces which are not all parallel are in equilibrium, their lines of action meet at a point, and they are capable of being represented by the sides of a triangle taken one way round. This is impossible in the case of a couple and a single force.

Hence, also, *no single force can be found to replace a couple.*

81. Equilibrium of Three Parallel Forces.—*If three parallel forces are in equilibrium, the middle force is in the opposite direction to each of the other two, and the three are proportional each to the perpendicular distance between the lines of action of the other two.*

Let the two outer forces be P and Q. Their resultant must be equal to the force R, and must act in the opposite direction along the same straight line. As the resultant of the two forces P and Q acts between them, these two forces must be in the same direction, so that their resultant is $P+Q$ in the same direction. Hence the middle force R must act in the opposite direction to each of the other two forces P and Q, and must be equal to their sum.

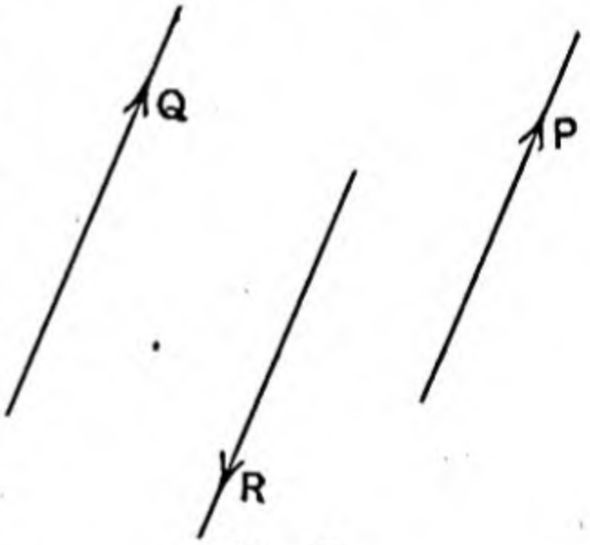


FIG. 76.

Also, if $(\overline{Q, R})$ denotes the perpendicular distance between the lines of action of Q and R, and similarly for the other pairs of forces, we have

$$P(\overline{R, P}) = Q(\overline{Q, R}).$$

$$\therefore \frac{P}{(\overline{Q, R})} = \frac{Q}{(\overline{R, P})} = \frac{P+Q}{(\overline{Q, R}) + (\overline{R, P})} = \frac{R}{(\overline{P, Q})}.$$

Thus the forces are proportional each to the perpendicular distance between the lines of action of the other two.

Example.—Fig. 77 represents a plank, of no appreciable weight, resting in a horizontal position upon two smooth supports at its extremities. A mass of weight W is placed upon the plank at a

point distant a and b from its extremities. It is required to determine the pressures P and Q between the plank and the supports.

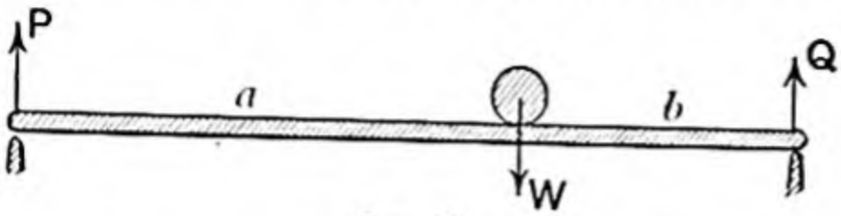


FIG. 77.

Consider the equilibrium of the plank and the mass as one system. The forces acting upon this system are the pressures P and Q of the supports, acting vertically upwards, and the weight W. The last-mentioned force is equal and opposite to and in the same straight line as the resultant of the other two.

Hence $$P+Q=W, \text{ and } Pa=Qb.$$

$$\therefore \frac{P}{b}=\frac{Q}{a}=\frac{P+Q}{b+a}=\frac{W}{a+b}.$$

$$\therefore P=\frac{b}{a+b}W \text{ and } Q=\frac{a}{a+b}W.$$

Otherwise.—The three forces P, Q, W are proportional each to the perpendicular distance between the other two.

$$\therefore \frac{P}{b}=\frac{Q}{a}=\frac{W}{a+b}$$

giving the same result as before.

EXAMPLES XII

1. Show that the resultant of two equal forces which act along parallel lines in the same direction, is double of each and acts midway between them.

2. Find the magnitude, direction, and position of the resultant of two parallel forces of 9 and 6 pounds' weight, acting respectively at points A and B, situated 30 inches apart, (i) when the forces are *like*, and (ii) when the forces are *unlike*.

3. Find the magnitude, direction, and position of the resultant of two parallel forces of 2 and 7 pounds' weight, acting respectively at points A and B, situated 45 inches apart, (i) when the forces are *like*, and (ii) when the forces are *unlike*.

4. Two parallel forces $5P$ and $7P$ act respectively at points A and B, situated one foot apart. Find the magnitude, direction, and position of their resultant (i) when the forces are *like*, and (ii) when *unlike*.

5. Two parallel forces, of 20 and 25 pounds' weight and of opposite senses, act on a rigid body, the perpendicular distance between their lines of action being 4 inches; find their resultant.

6. A straight line $OBCA$ is drawn to intersect the lines of action of two like parallel forces of P and Q pounds' weight in the points A and B respectively, and the line of action of their resultant R pounds' weight in C. The distances OA, OB, OC being of lengths a, b, c inches respectively, prove that $R=P+Q$ and $Rc=Pa+Qb$. Also determine the unknown quantities in each of the following cases:—

$\frac{P}{b-c}=\frac{a}{a}=\frac{Q}{c-a}=\frac{b}{b}=\frac{R}{b-a}=\frac{c}{c}$ (Art. 81)

Or $P(c-a)=Q(b-c)$ [Art. 76] $\therefore Pa+Qb=P(c+a-c-a)+Q(c+b-c)=(P+Q)c=Rc$.

(i) Given $P=5$, $Q=3$, $a=6$, $b=4$, find R and c.
(ii) Given $P=6$, $R=9$, $a=6$, $b=0$, find Q and c.
(iii) Given $P=9$, $Q=8$, $a=20$, $c=12$, find R and b.
(iv) Given $P=8$, $R=14$, $a=15$, $c=9$, find Q and b.
(v) Given $a=10$, $b=5$, $c=7$, find P and R in terms of Q.
(vi) Given $a=10$, $b=2$, $c=5$, find P and Q in terms of R.

7. A straight line $OBAC$ is drawn to intersect the lines of action of two unlike parallel forces of P and Q pounds' weight in the points A and B respectively, and the line of action of their resultant R pounds' weight in C. The distances OA, OB, OC being of lengths a, b, c inches respectively, prove that $R=P-Q$ and $Rc=Pa-Qb$. Also determine the unknown quantities in each of the following cases :—

(i) Given $P=6$, $Q=5$, $a=9$, $b=4$, find R and c.
(ii) Given $Q=3$, $R=5$, $a=10$, $b=0$, find P and c.
(iii) Given $P=11$, $Q=6$, $a=4$, $c=7$, find R and b.
(iv) Given $P=25$, $R=16$, $a=7$, $c=9\frac{1}{4}$, find Q and b.
(v) Given $a=8$, $b=5$, $c=14$, find Q and R in terms of P.
(vi) Given $a=3$, $b=0$, $c=5$, find P and Q in terms of R.

8. A force of 20 pounds' weight acts at B and a parallel force of P pounds' weight acts at A in the opposite direction. The line of action of the resultant of these two forces intersects BA produced at C. If the force at A is diminished by 8 pounds' weight, the line of action of the resultant intersects AB produced at C'. If A is the middle point of CC', find the value of P.

9. Assuming the magnitude, direction, and position of the resultant of two *like* parallel forces, deduce the magnitude, direction, and position of the resultant of two *unlike* parallel forces.

10. If a heavy body is partly supported by a string and partly by a smooth horizontal plane, prove that the string must be vertical.

11. A uniform rod, 2 feet long and weighing 3 pounds, lies on a horizontal plane; find the least force which, applied 5 inches from one end, will raise that end above the plane.

12. Two workmen are carrying a heavy ladder on their shoulders; find the portions of the weight of the ladder supported by them, the centre of gravity of the ladder being at a distance of 3 feet from one workman and 6 feet from the other.

13. A horizontal bar AB, 7 feet long, is supported at its extremities, and a man of 150 pounds' weight hangs from it by his hands, one being 1 foot from A, the other 3 feet from B. Find the pressures on the supports due to the weight of the man.

14. A heavy pole, weighing 140 pounds, is carried on the shoulders of two men, one at each end; the centre of gravity of the pole being 2 feet from one end and 5 feet from the other, find the loads supported by the men.

Also, find what would be the effect of placing each man one foot nearer to the centre of gravity of the pole.

15. A heavy uniform rod AB, of mass 12 ounces, is suspended from a fixed point by means of a fine light string attached to the rod at P, such that AP is one-third of AB. Find what mass, suspended from A, will keep the rod horizontal.

If the rod is then tilted round P into a different position in a vertical plane, will the equilibrium be disturbed?

16. A fine rod, of no appreciable weight, is loaded with a mass of weight W, and is supported in a horizontal position upon two smooth pegs A and B, situated at a distance c apart. If the pegs A and B cannot withstand pressures greater than P and Q respectively, show that the load must be applied at some point of the rod situated within a certain portion of length

$$\frac{P+Q-W}{W}c.$$

17. In the preceding example, determine where the load must be applied, in order that the pressure at A may be as much less than P as the pressure at B is less than Q.

18. A rectangular portmanteau, 3 feet in length and 2 feet in height, and weighing 56 pounds, is carried up a staircase by two men supporting it along the front and back edges of its bottom face. If this face is held at an inclination of 30° to the horizontal, find what portions of the weight are supported by the two men, supposing the centre of gravity of the portmanteau to be at its centre of figure.

82. Effect of a Force in Producing Rotation about a Fixed Point.—Suppose that a body is capable of turning freely about a fixed point O. The constraint at O will produce a self-adjusting force which accommodates itself to prevent the point O of the body from getting away from the point O of space, if possible.

If a force P is applied to the body along a line which passes through O, then, assuming that the connection at O is strong enough to withstand all stress upon it, the constraint at O will resist the effect of the force P to move the body by exerting an equal force P in the opposite direction, and the body will be in equilibrium. Thus the force P has no tendency to produce a rotation of the body about O.

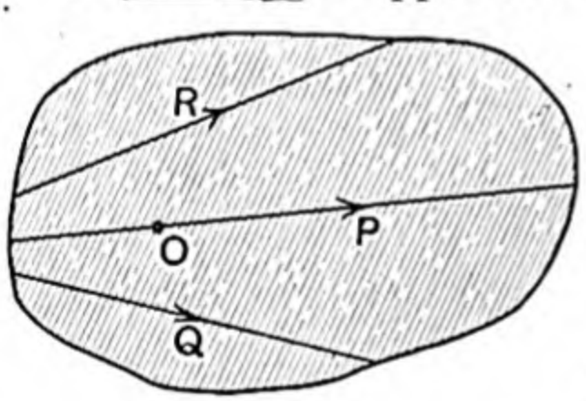


FIG. 78.

A force, such as Q or R, whose line of action does not pass through O, cannot alone be balanced by the constraint at O, and would set up a rotation of the body about O.

In Fig. 78 the forces Q and R would tend to produce rotations in opposite directions. One direction is chosen as the positive direction of rotation, then the opposite direction is the negative direction of rotation.

Now suppose that a body, capable of turning freely about O,

is acted upon by two forces P and Q in addition to the force of constraint at O. Let p and q be the lengths of the perpendiculars from O upon the lines of action of the forces P and Q respectively.

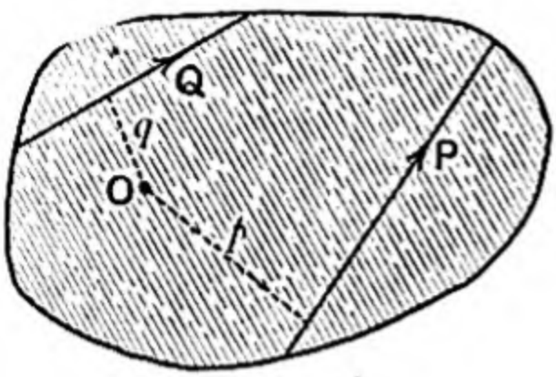


FIG. 79.

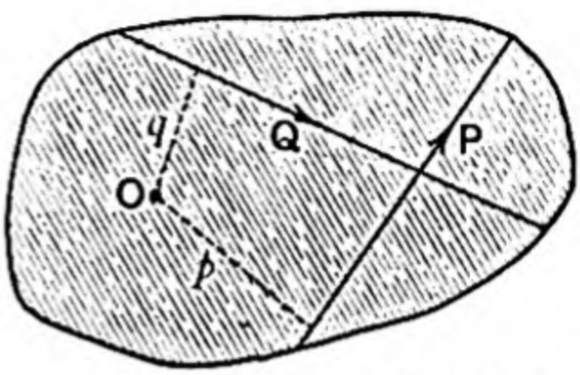


FIG. 80.

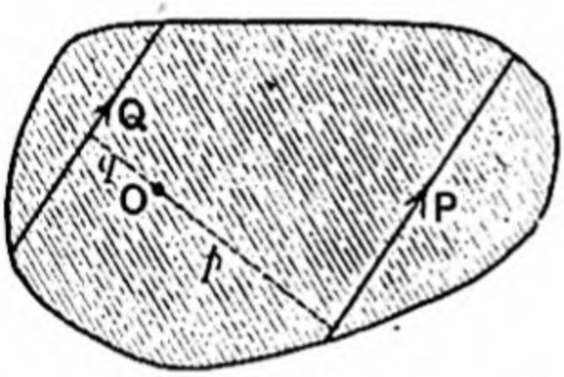


FIG. 81.

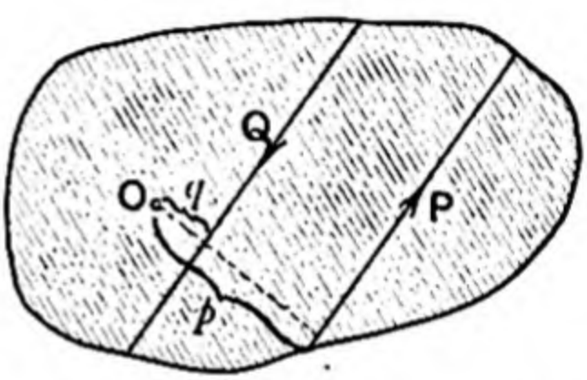


FIG. 82.

Then, if the body is in equilibrium, *either* the two forces P and Q must be equal and opposite and act along the same straight line (in which case there is no constraint at O and the body would be in equilibrium even if O were not fixed), *or* the two forces P and Q must be equivalent to a single force which acts along a line through O (in which case the constraint at O produces an equal force acting in the opposite direction along the same straight line).

In the *first* case, we have $P = Q$ and $p = q$, and the forces tend to produce rotations in opposite directions about O.

In the *second* case, we have by Arts. 29 and 78, $Pp = Qq$, and the forces tend to produce rotations in opposite directions about O.

Thus, in *both* cases, $Pp = Qq$, and the forces tend to produce rotations in opposite directions about O.

Conversely, if $Pp = Qq$, and if the forces tend to produce rotations in opposite directions about O, then, by Arts. 29 and 78, they will be equivalent to a single force whose line of action passes through O, so that the body will be in equilibrium.

83. It appears from the above that the force P might be replaced by any other force P', which tends to produce rotation in the same direction about O, and which is such that the length of the perpendicular from O upon its line of action is p', where $Pp = P'p'$, and the body would still be in equilibrium.

Thus, the tendency of a force to produce rotation about a fixed point is properly measured by the product of the measure of the force and the measure of the perpendicular distance of its line of action from the fixed point.

84. Moment of a Force about a Point.—The *moment* of a force about a point is the tendency it would have to produce a rotation of the body about the point, if that point were to become fixed.

The *arm* of a force with respect to a point is the perpendicular distance of its line of action from the point.

The moment of a force about a point is *measured* by the product of the measure of the force and the measure of its arm with respect to the point.

The *sign* of the moment is + or − according as the direction in which it tends to produce rotation about the point is in the direction chosen as positive or in the opposite direction.

Thus, in the figures of Art. 82, if the positive direction of rotation is chosen to be contrary to that in which the hands of a clock revolve, then the moment of the force P about O is given by $+Pp$, and the moment of the force Q is given by $-Qq$. The sum of these two moments is zero, and the writing down of the equation $Pp - Qq = 0$ is described as *taking moments about O*.

[The student should notice that the moment of a force vanishes only when its line of action passes through the point about which its moment is estimated.]

85. Principle of Moments.—*The sum of the moments of any two forces about any point in their plane is equal to the moment of their resultant about that point.*

Case I.—Let the two forces P and Q act along lines which intersect at A. Let R be their resultant, which also acts along a line through A.

Let O be the point about which moments are to be taken. Join OA.

Draw AX at right angles to OA in such a direction, that if the body were to commence to rotate round O in the direction

chosen as positive, then the point A would begin to move towards X.

Let P_1, Q_1, R_1 be the resolved parts of P, Q, R respectively in the direction AX. Then we know that $P_1 + Q_1 = R_1$.

Now Take AB to represent P, and draw ON and BM perpendiculars upon AB and AX respectively.

Let p and a be the lengths of ON and OA respectively.

The angle ABM = the angle OAN.

$\therefore AM : AB = ON : OA = p : a$,

$\therefore a(AM) = p(AB) = pP$.

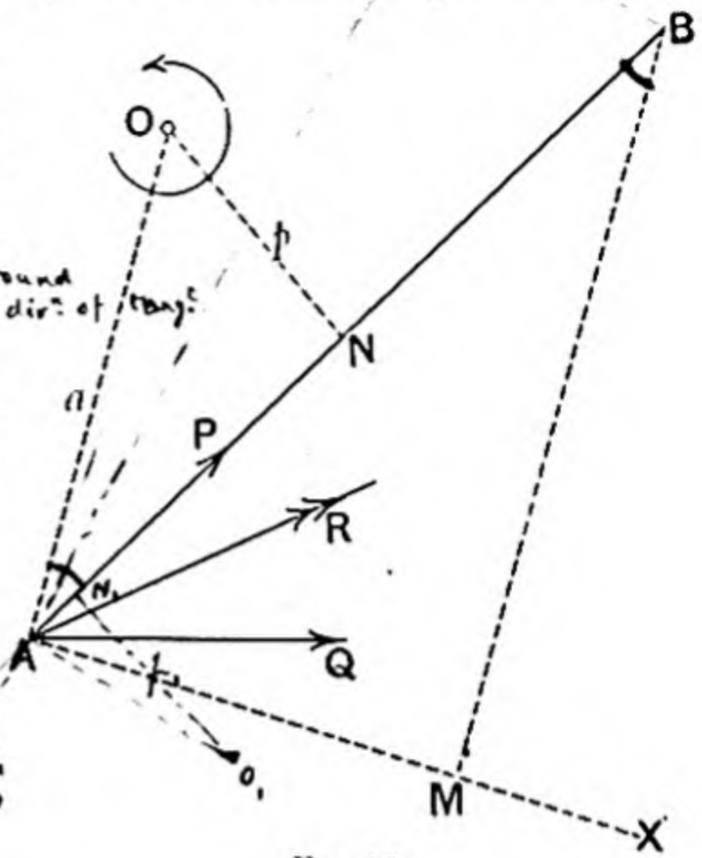


FIG. 83.

Now the moment of P about O is $+pP$ or $-pP$ according as the force tends to produce rotation about O in the positive direction or in the negative direction, that is according as AM lies along AX or along XA produced.

Therefore the moment of P about O is $a(+AM)$ or $a(-AM)$ according as AM lies along AX or along XA produced.

But P_1 is represented by AM, and is positive or negative according as AM lies along AX or along XA produced.

Thus, the moment of P about O is given by aP_1 both in magnitude and sign.

Similarly, the moments of Q and R are given respectively by aQ_1 and aR_1 both in magnitude and sign.

Therefore

Sum of moments of P and Q about O

$$= aP_1 + aQ_1$$
$$= a(P_1 + Q_1)$$
$$= aR_1$$
$$=$$ moment of R about O.

Case II.—Let the two forces P and Q be parallel.

Then, with the construction and figures of Art. 78, we may replace the force P by two forces represented by AO and OB acting at M, and the force Q by two forces represented by BO and OC acting at N.

By Case I., the moment of P about any point is equal to the sum of the moments of its components about that point, and similarly for the force Q.

Therefore

Sum of moments of P and Q about any point

= moment of force represented by AO acting at M
+ " " " " OB " " M
+ " " " " BO " " N
+ " " " " OC " " N
= moment of force represented by AO " " M
+ " " " " OC " " N
= moment of resultant about the same point.

In the case of a couple (see figures of Art. 79).

Sum of moments of the two forces P [Put A for C in the above + omit last line.]

= sum of moments of the two forces P'.

[Cf. Art 84] The student should notice that the sum of the moments of two forces vanishes when the line of action of their resultant passes through the point about which the moments are estimated, and also when the forces are equal and act in opposite directions along the same straight line, and under no other circumstances.

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Case II. may be proved in the manner suggested in questions 6 and 7 of Examples XII. i.e. by Art. 84 (or 785. Loney.)

86. Moment of a Couple.—*The sum of the moments of the two forces which constitute a couple is the same about any point in the plane of the couple.*

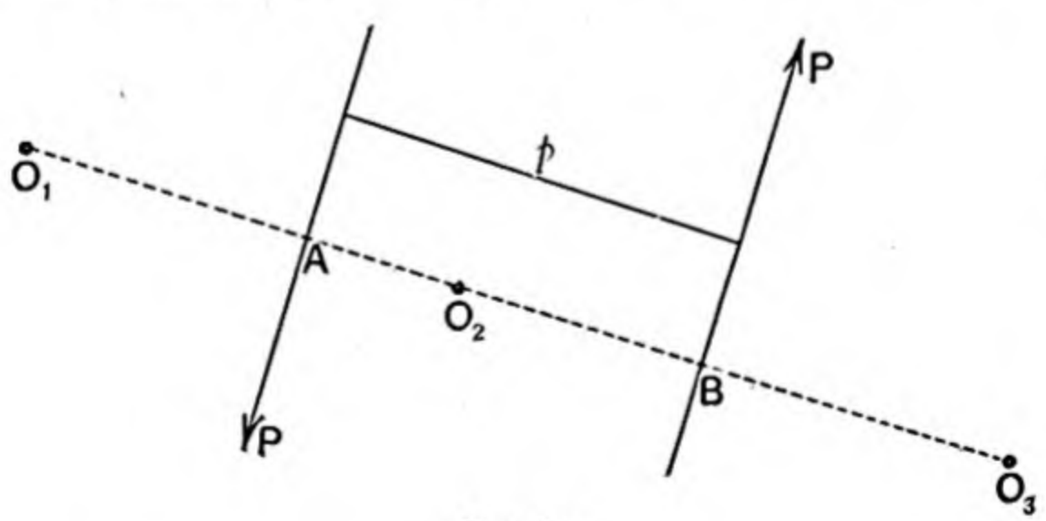

FIG. 84.

Let the couple consist of two forces each of magnitude P, and let p be the length of the perpendicular distance between their lines of action.

Let O be any point in the plane of the couple, and let the

straight line drawn through O perpendicular to the direction of the two forces meet their lines of action in A and B.

Then, if O is in BA produced, as at O_1,
Sum of moments of the forces about O

$$= P(O_1 B) - P(O_1 A)$$
$$= P(AB)$$
$$= Pp.$$

If O is between A and B, as at O_2,
Sum of moments of the forces about O

$$= P(O_2 A) + P(O_2 B)$$
$$= P(AB)$$
$$= Pp.$$

If O is in AB produced, as at O_3,
Sum of moments of the forces about O

$$= P(AO_3) - P(BO_3)$$
$$= P(AB)$$
$$= Pp.$$

The perpendicular distance between the lines of action of the two forces which constitute a couple is called the *arm* of the couple.

The sum of the moments about any point of the two forces which constitute a couple is called the *moment of the couple*, and is of constant magnitude for all positions of the point, being measured by the product of the measure of one of the forces and the measure of the arm.

87. Lever.—A *lever* consists essentially of a rigid body capable of turning freely about a fixed point called the *fulcrum*. It is usually in the form of a rigid bar, which may be straight or bent or curved. A bent lever, unless the contrary is stated, is understood to consist of two straight portions with the fulcrum situated at the bend.

The lever is used for the purpose of overcoming a resistance applied at one point of it by means of a force applied at another point. The resistance to be overcome is usually called the "Weight," and the force applied to the lever to overcome the "weight" is usually called the "Power."

Let O be the fulcrum and let the "Power" P and the "Weight" W be applied at points A and B respectively. The parts OA and OB are called the arms of the lever.

We will suppose that the lever is of no appreciable weight,

and that it is in equilibrium, the only forces acting upon it being the forces P and W and the resistance R of the fulcrum.

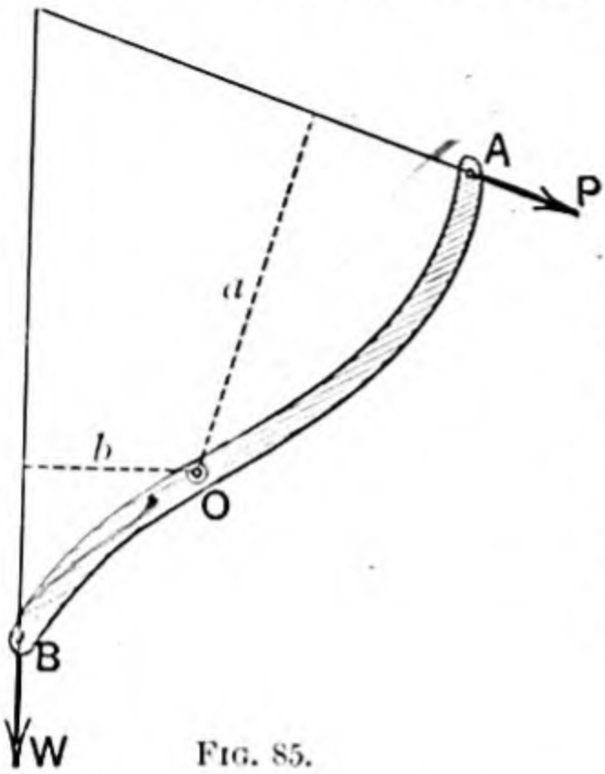

FIG. 85.

Let a and b be the lengths of the perpendiculars from the fulcrum upon the lines of action of the forces P and W respectively. Then, if the connection at O is strong enough to withstand all stress upon it, we have seen in Art. 82 that it is sufficient for the equilibrium of the lever that

$$Pa = Wb.$$

This result is often quoted as *The Principle of the Lever*, and shows that $\frac{W}{P}$, which is called the *mechanical advantage*, is equal to $\frac{a}{b}$.

If it is required to determine the resistance R of the fulcrum, we may suppose the fulcrum to be removed and the force R applied at the point O, so that the lever becomes a rigid body free to move and in equilibrium under the action of the three forces P, W, and R. The consideration of the equilibrium comes under the methods of Chapter III., and the line of action of the force R is the line joining O to the point of intersection of the lines of action of P and W.

If the forces P and W are parallel, then the resistance R of the fulcrum is a third parallel force.

88. Classification of Levers.—In the most simple form, the arms of a lever are in one and the same straight line, and the forces P and W are parallel.

It is usual to divide levers into classes according to the positions of the points of application of the "power" and the "weight" relatively to the fulcrum.

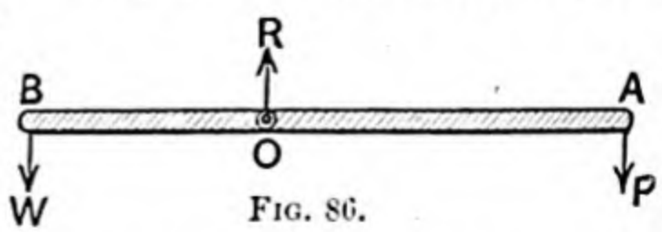

FIG. 86.

In levers of the *first* class the "power" and the "weight" are applied on opposite sides of the fulcrum, as in Fig. 86.

In this case the forces P and W are in the same direction, and the reaction R of the fulcrum is in the opposite direction and equal to $P + W$.

As examples we have a *pump-handle*, a *poker*; while a *pair of scissors* constitutes a double lever of this class.

In levers of the *second* class the "power" and the "weight" are applied on the same side of the fulcrum, the "power" being applied at a greater distance from the fulcrum than the "weight."

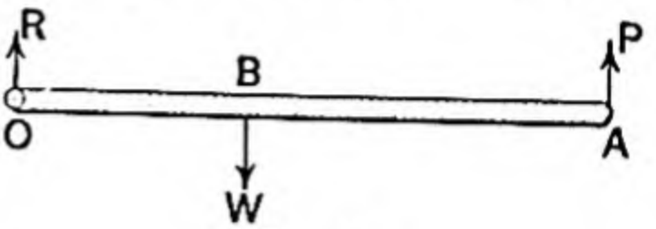

FIG. 87.

In this case the forces P and W are in opposite directions, and the reaction R is in the direction of P and equal to $W - P$.

As examples we have a *cork squeezer*, a *wheel-barrow*; while a *pair of nutcrackers* constitutes a double lever of this class.

In levers of the *third* class the "power" and the "weight" are applied on the same side of the fulcrum, the "power" being applied nearer to the fulcrum than the "weight."

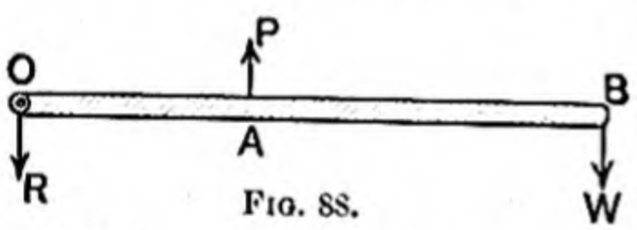

FIG. 88.

In this case the forces P and W are in opposite directions and the reaction R is in the direction of W and equal to $P - W$.

As examples we have *the treadle of a lathe*, *the bone of the human forearm* when bent at the elbow and supporting a load in the hand; while a *pair of sugar-nippers* constitutes a double lever of this class.

In all three cases, we have $P(OA) = W(OB)$, and the *mechanical advantage* equals $\frac{OA}{OB}$.

89. Pulley.—A *pulley* consists of a circular wheel, capable of turning in its own plane about an axis through its centre. The axis on which the pulley turns is mounted in a separate piece called the *pulley-block*, or simply the *block*. The block and the pulley are shown separated in Fig. 90.

The circumference of the pulley is grooved so as to receive a string, and is sufficiently rough to prevent the string from slipping over its surface.

The pulley is said to be smooth when it can turn freely about its centre.

The term *pulley* is often applied to the whole arrangement consisting of pulley and block.

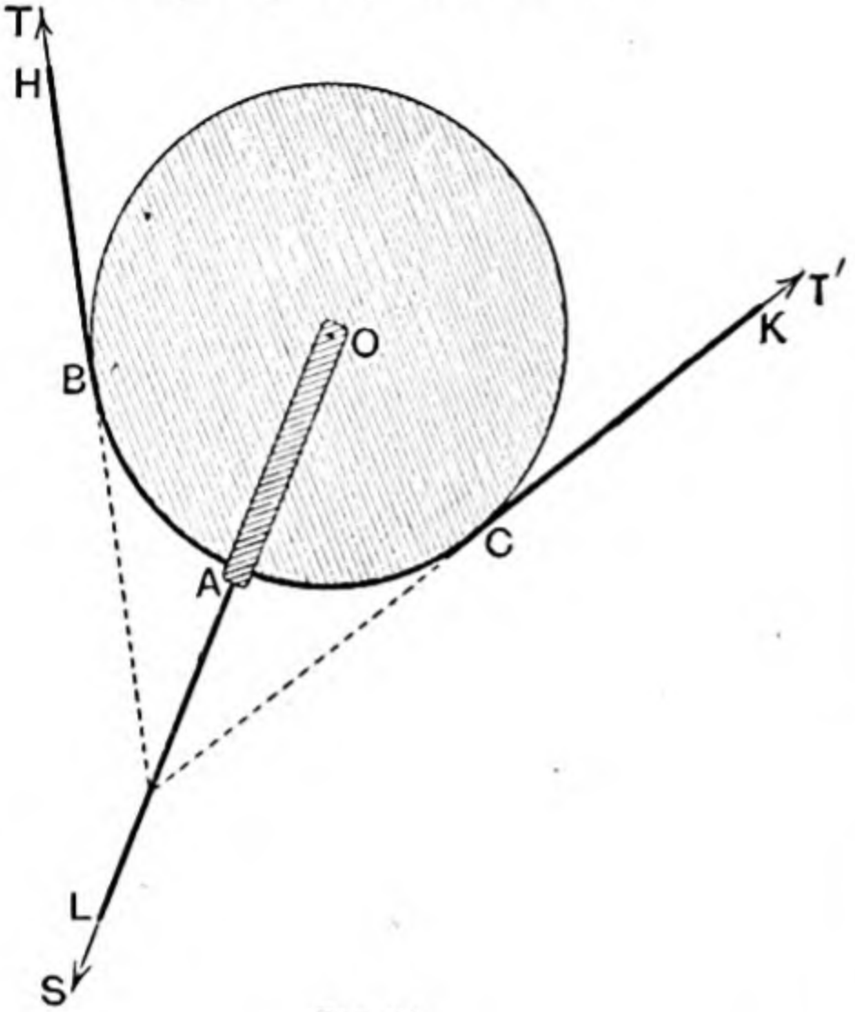

FIG. 89.

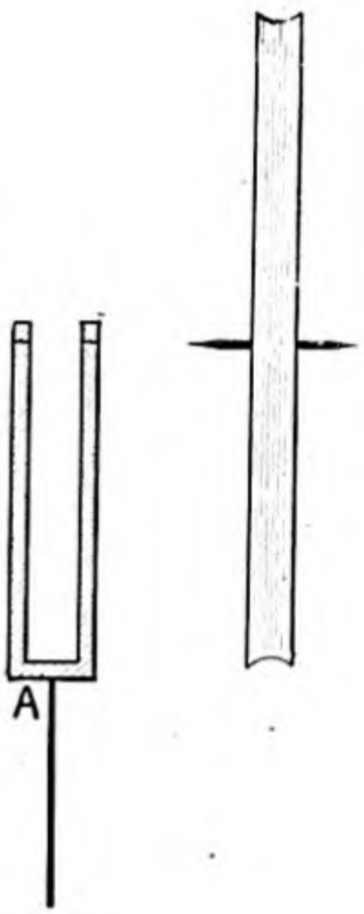

FIG. 90.

The pulley is said to be fixed when the block is fixed, and is said to be movable when the block is movable.

When in use, a string *HBCK* in a state of tension passes round a portion *BC* of the circumference of the pulley, and a string *AL*, also in a state of tension, is attached to the block at *A*.

We will suppose that the weight of the pulley and block is inappreciable, and that the pulley is smooth; also that the strings are fine and light, so that *AL*, *BH*, *CK* are straight, and *BH* and *CK* are the tangents at *B* and *C* respectively.

Let *O* be the centre of the pulley, and, when there is equilibrium, let *S*, *T*, *T′* be the tensions of *AL*, *BH*, *CK* respectively.

Considering the equilibrium of the system consisting of the string *HBCK* together with the pulley apart from the block, we see that the moments about *O* of the forces *T* and *T′* are equal and of opposite sign; and, as the perpendiculars from *O* upon their lines of action are equal, it follows that $T = T'$.

Considering the equilibrium of the whole system shown in Fig. 89, we see that either the strings are all parallel or they

meet in a point, and the force S is equal and opposite to the resultant of the two equal forces T.

If the strings are all parallel, we have

$$S = 2T.$$

If the strings BH and CK are inclined at an angle θ, the line AL bisects that angle and

$$S = 2T \cos \tfrac{1}{2}\theta.$$

Considering the equilibrium of the block alone without the pulley, we see that the line AL passes through the centre of the pulley, and the pressure between the pulley and the axle is of magnitude S and in the line LAO.

***90.** Ex. 19.—*A rigid body is capable of turning freely in one plane about a fixed point A, and is acted upon by a given force P. An unknown force X is applied at a given point B of the body. It is required to determine the different values, corresponding to different directions, of X consistent with equilibrium.*

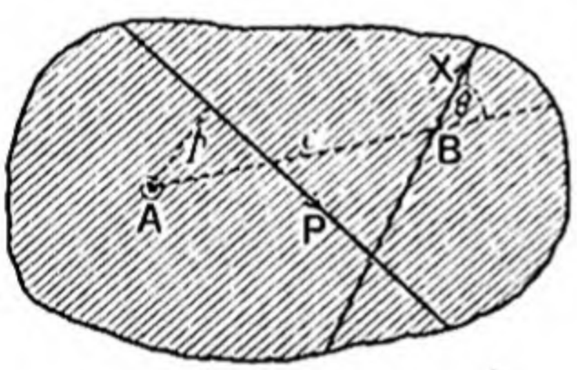

FIG. 91.

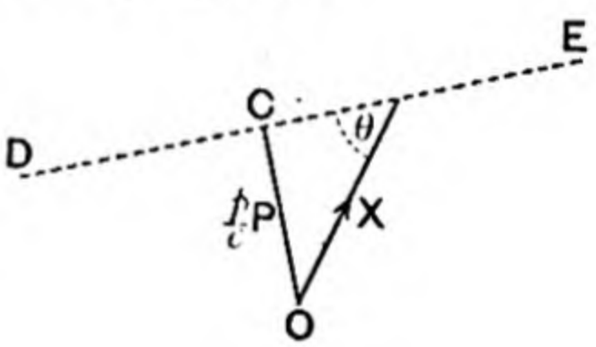

FIG. 91a.

Let the force X make an angle θ with AB, so that θ is variable. Let AB be of given length c; then the perpendicular from A upon the line of action of the force X is of length $c \sin \theta$.

The two forces X and P must have equal and opposite moments about A. Hence, if p is the known length of the perpendicular from A upon the line of action of the given force P, we have

$$X \cdot c \sin \theta = Pp,$$

$$\therefore X \sin \theta = \frac{Pp}{c} = K \quad \therefore \sin\theta = \frac{K}{X} \quad (2 \text{ vals. of } \theta)$$

Thus the resolved part of the force X in a direction perpendicular to AB is constant and equal to $\frac{Pp}{c}$.

From a fixed point O draw OC in a direction perpendicular to AB and of length $\frac{p}{c}P$ units, and draw DCE through C parallel to AB. Then the different values of X are represented by the different straight lines drawn from O to different points in the line DE.

The shortest line from O to DE is the perpendicular OC. Hence X is smallest when it is applied in a direction perpendicular to AB.

*91. Ex. 20.—*Two light rods ABD and ACE, of lengths d and e respectively, can turn freely in a vertical plane about a common fulcrum A, and the points B and C, situated at distances c and b respectively from A, are connected by a fine light string of length a. It is required to determine the least force which, applied at E, will keep AD horizontal when a mass of given weight W is suspended from D; also to find the tension of the string.*

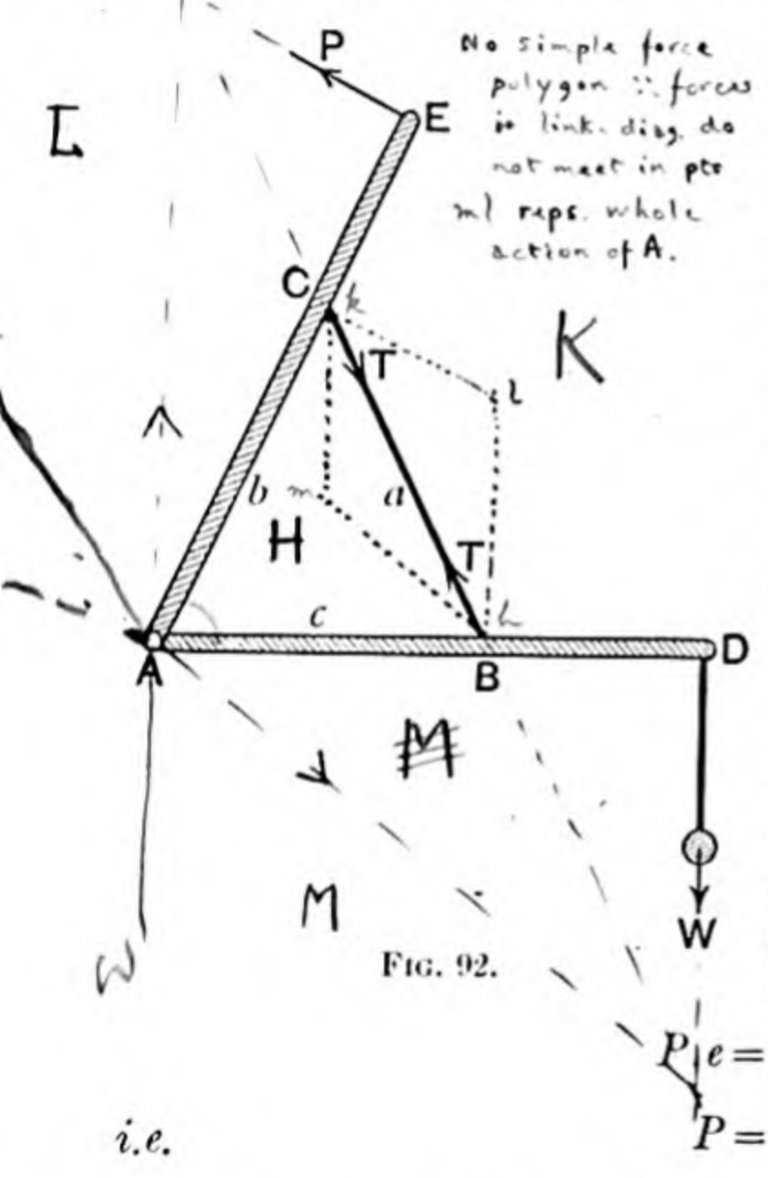

FIG. 92.

Taking moments about A for the equilibrium of the whole system, we see that the force at E is least when it is applied in a direction perpendicular to AE, and its value is then P, where

$$P.e = W.d,$$

i.e.

$$P = W.\frac{d}{e}.$$

The perpendicular from A upon the string is of length $\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}$, where $2s = a+b+c$. Hence, if the tension of the string is of magnitude T, taking moments about A for the equilibrium of the rod AD alone, we have

$$T.\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a} = Wd.$$

Thus $$T = \frac{W \cdot ad}{2\sqrt{s(s-a)(s-b)(s-c)}}.$$

92. Ex. 21.—*A heavy rod ACDB rests in a horizontal position upon two smooth pegs C and D, situated in given positions. The greatest load which can be applied at A without disturbing the equilibrium is of given weight P, and the greatest load which can be applied at B without disturbing the equilibrium is of given weight Q. It is required to determine the weight of the rod, and the position of its centre of gravity.*

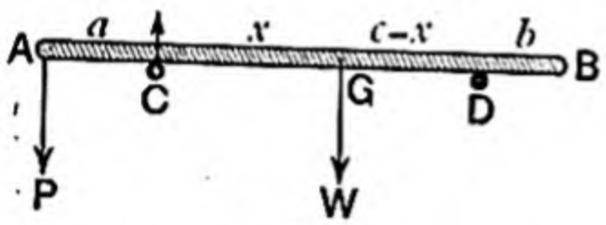

FIG. 93.

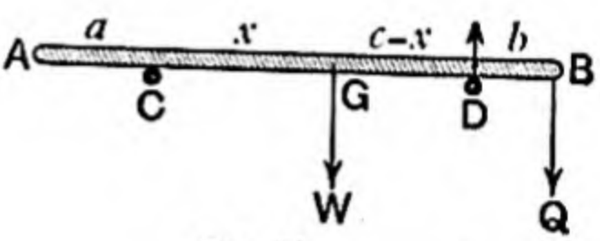

FIG. 94.

Let AC, CD, DB be of lengths a, c, b respectively. Let the weight of the rod be of magnitude W, and let its centre of gravity G be at a distance x from the peg C, W and x being at present unknown.

When the load is applied at A, the rod is on the point of turning about C, and there is no pressure between the rod and the peg D. Hence the moments of P and W about C must be equal and opposite.

i.e. $$Wx = Pa.$$

When the load is applied at B, the rod is on the point of turning about D, and there is no pressure between the rod and the peg C. Hence the moments of Q and W about D must be equal and opposite.

i.e. $$W(c-x) = Qb.$$

Adding this result to the equation previously obtained, we have

$$Wc = Pa + Qb,$$

$$\therefore W = \frac{Pa + Qb}{c}.$$

Hence, from the first equation,

$$x = \frac{Pa}{W} = \frac{Pac}{Pa + Qb}.$$

***93.** Ex. 22.—*A straight rod AB, of no appreciable weight, is fixed in the ground at A, but can be broken off at A when a force is applied to it by means of a light cord. The cord being of given length, it is required to determine where it must be attached, in order that the rod may be overturned with the least exertion by a force applied to that extremity of the cord which is on the ground.*

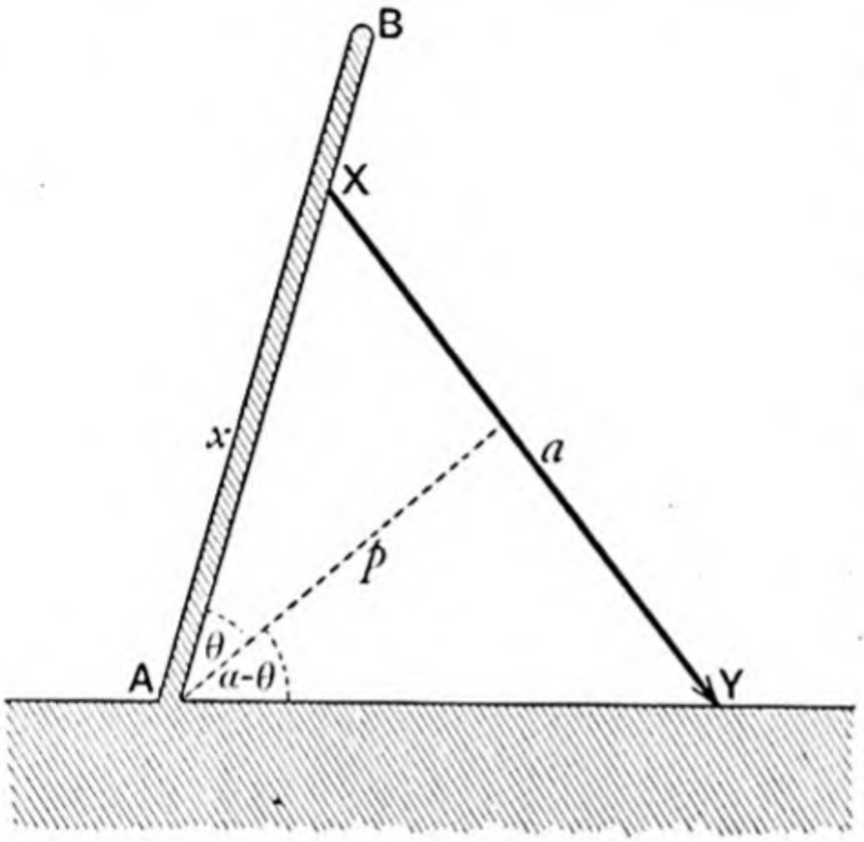

FIG. 95.

Let the cord XY be of given length a, and let it be attached to the rod at a point X distant x from A. Let the angle BAY be of known magnitude α, and let the perpendicular from A upon XY be of length p and make angle θ with AB. Thus x, p and θ are all unknown, but we can express x and p each in terms of θ.

We have

$$\frac{x}{\sin AYX} = \frac{a}{\sin XAY},$$

$$\therefore x = \frac{a \cos(\alpha - \theta)}{\sin \alpha},$$

and

$$p = x \cos\theta = \frac{a \cos(\alpha - \theta) \cos\theta}{\sin \alpha}$$

$$= \frac{a}{2 \sin \alpha}\{\cos \alpha + \cos(\alpha - 2\theta)\}.$$

Now the rod breaks off at A as soon as the force applied at Y has a moment about A sufficiently great. Hence the rod is overturned with the *least* exertion, when the value of p is *greatest*.

Thus $\cos(a - 2\theta)$ must be a maximum,

$$\therefore a - 2\theta = 0,$$

$$\therefore \theta = \tfrac{1}{2}a$$

and
$$x = \frac{a \cos \frac{1}{2}a}{\sin a} = \frac{a}{2 \sin \frac{1}{2}a}.$$

Otherwise.—As the length of XY and the magnitude of the angle XAY are both given, the segment of the circle described upon XY and passing through A is fixed in size though not in position.

Now the greatest perpendicular from a point situated in the arc of a segment upon the base of the segment, is the perpendicular which bisects the base.

Thus the perpendicular from A upon XY must bisect XY.

$\therefore AX = AY$, and we have the same result as before.

***94.** Ex. 23.—*A rectangular block, of length a and height b, is being turned over by a smooth crowbar of length c, applied at the middle point of the lower edge of one end of the block in a vertical plane through its centre of figure. Find what force must be applied at right angles to the crowbar at one end, so that it may rest inclined at an angle α to the horizontal, the lower face of the block being then inclined at an angle β to the horizontal. The block is of given weight W, and its centre of gravity coincides with its centre of figure.*

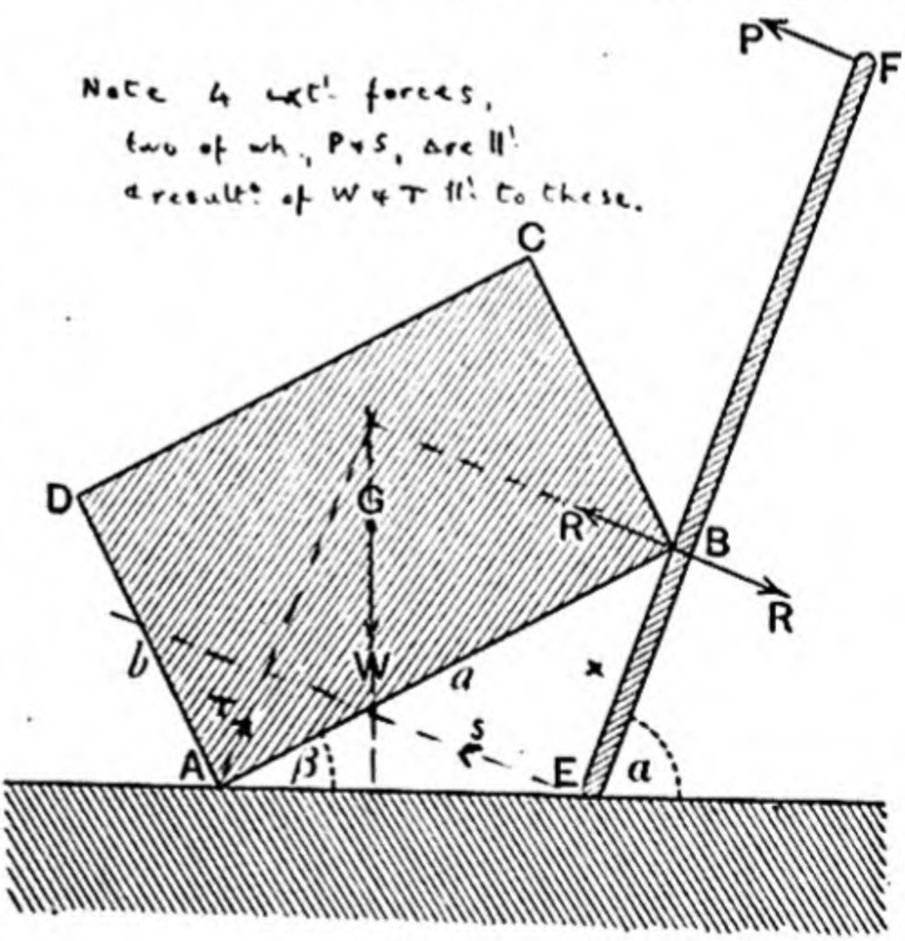

FIG. 96.

Let $ABCD$ represent the vertical section of the block through its centre of figure G, and let it rest against the ground at A and against the crowbar at B.

Let P be the force applied to the crowbar at F, and, regarding

the surfaces in contact at B as smooth, let R be the pressure between the block and the crowbar in a direction perpendicular to EF.

(Trig.) If EB is of length x, we have

$$\frac{x}{\sin\beta} = \frac{a}{\sin\alpha},$$

$$\therefore\ x = \frac{a\sin\beta}{\sin\alpha}.$$

Taking moments about E for the equilibrium of the crowbar, we have

$$R.x = P.c$$

$$\therefore\ R = \frac{Pc\sin\alpha}{a\sin\beta}.$$

Art. 85 We now take moments about A for the equilibrium of the block. The moment of the force W is most readily determined by adding together the moments of its two components, namely, $W\cos\beta$ perpendicular to AB and $W\sin\beta$ parallel to BA. In determining the moment of the force R, we notice that the perpendicular from A upon its line of action is of length $a\cos ABE = a\cos(\alpha-\beta)$.

Thus, we obtain

$$\frac{a}{2}.W\cos\beta - \frac{b}{2}W\sin\beta = R.a\cos(\alpha-\beta)$$

$$= \frac{Pc\sin\alpha\cos(\alpha-\beta)}{\sin\beta},$$

$$\therefore\ P = \frac{(a\cos\beta - b\sin\beta)\sin\beta}{2c\sin\alpha\cos(\alpha-\beta)}.W.$$

EXAMPLES XIII

1. A and B are two fixed points in a given plane, situated 10 inches apart. A force P acts through A, but has any direction whatever in the plane. If P has always a moment of 60 inch-pounds'-weight about B, give a graphic representation of the various magnitudes and directions of P.

What is the least, and what the greatest, value of P?

2. The sides BC, CA, AB of a triangle are 3, 4, and 5 feet respectively; find the magnitude and direction of a force at C whose moments about A and B are 7 and 5 foot-pounds'-weight respectively.

3. The side BC of a triangle ABC is bisected at E and produced to F. Show that the sum of the moments about F of the forces which act along and are represented by AB and AC is equal to twice the

moment about F of the force which acts along and is represented by AE.

4. P is a fixed point on the circumference of a fixed circle; PM and PN are any two chords of the circle at right angles to one another; Q is any other fixed point whatever in the plane of the circle. Show that if two forces act along and are represented by PM and PN, then the algebraic sum of their moments about Q is constant.

5. OA and OB are chords, 4 and 5 inches in length, of a circular disk $OACB$, whose diameter OC is 6 inches. If forces of 3 and 4 pounds' weight act from O along these chords respectively, find how the disk will begin to move, the point C being fixed.

6. $ABCD$ is a square disk lying on a smooth horizontal plane, and capable of turning freely about the point A, which is fixed. At E, the middle point of AB, a force of 16 pounds' weight acts in direction CE, and at D a force of 10 pounds' weight acts in direction BD. Find how the disk will begin to move.

7. $ABCDEF$ is a thin board in the form of a regular hexagon, and capable of turning freely on a smooth horizontal table about the point A, which is fixed. A force of 50 pounds' weight is applied at B in direction DB, and another of 33 pounds' weight is applied at E in direction CE. Determine in what direction the board moves.

8. A straight lever AOB, of no appreciable weight, balances about O when loaded at A and B. If each load is increased by the same amount, show that the extremity of the longer arm descends.

9. A straight lever AOB, of no appreciable weight, is capable of turning freely about a fulcrum O, and is loaded at A and B. Determine the unknown quantities in each of the following cases, in order that there may be equilibrium—

(i) The lever is of length one foot, and masses of 5 and 3 pounds are suspended from A and B respectively; find the length of AO.

(ii) AO is of length 6 inches, and masses of 8 and 12 pounds are suspended from A and B respectively; find the length of the lever.

(iii) The lever is of length $2\frac{1}{2}$ feet, and the pressure on the fulcrum is 60 pounds' weight, the load at A being 36 pounds; find the length of AO.

(iv) AO is of length 10 inches, and the pressure on the fulcrum is 48 pounds' weight, the load at A being 28 pounds; find the length of the lever.

(v) AO and OB are of lengths 12 and 16 inches respectively; find the pressure on the fulcrum, the load at A being 20 pounds.

(vi) AO and OB are of lengths 8 and 10 inches respectively; find the loads applied at A and B, the pressure on the fulcrum being 63 pounds' weight.

10. A straight lever, of length a and of no appreciable weight, balances about one point when masses of weight P and W are suspended from its extremities. If W is increased by w and P diminished by the same amount, show that the fulcrum must be moved through a distance $\frac{wa}{P+W}$.

11. A straight lever, of no appreciable weight, has its fulcrum at one extremity, and the "Weight" is applied at a distance a from that extremity. If the pressure upon the fulcrum must not exceed nW in either direction, upwards or downwards, where n is a given proper fraction, show that the "Power" must act somewhere within a distance equal to $\frac{2na}{1-n^2}$.

12. A straight light rod, of length a, rests in a horizontal position between two fixed pegs, placed at a distance c apart, one of the pegs being at one end of the rod. Find the pressures on the pegs when a mass of weight W is suspended from the other end.

13. A heavy uniform bar has centre C and one end D. It is placed across two pegs A and B, which are in the same horizontal straight line, in such a manner that they divide the distance CD symmetrically, and it is kept in equilibrium by a mass of weight P suspended from D. If the largest possible value of P is λ^2 times the smallest possible value, show that the distance between the pegs is $\frac{1}{2}\cdot\frac{\lambda-1}{\lambda+1}$ of the length of the bar.

14. A uniform straight plank $APQB$, 12 feet long, rests upon two supports at P and Q. The plank is just on the point of tilting up when a man of twice its weight stands at B;—also when he stands at the middle point of AP. Find the positions of P and Q.

15. In the preceding question, find where the man must stand, in order that the pressures on the supports may be equal to one another.

16. A heavy uniform bar $ACDB$ rests in a horizontal position upon two fixed supports C and D, whose distance apart is 6 inches, and equal to the length of the projecting part AC of the bar. If an upward force of two pounds' weight, applied at A, just lifts the bar off the support C, and a downward force of 8 pounds' weight at A just lifts it off D, find the length and weight of the bar.

17. A rod ABC, 16 inches long, rests in a horizontal position upon two supports at A and B, one foot apart, and it is found that the least upward and downward forces applied at C, which would move the rod, are 4 ounces' weight and 5 ounces' weight respectively. Find the weight of the rod, and the position of its centre of gravity.

18. A heavy straight rod ABB_1A_1, of length $a+b+a_1$, rests in a horizontal position upon two smooth pegs B and B_1, distant a and a_1 from A and A_1 respectively. The greatest loads which can be applied in turn at A and A_1, without disturbing the equilibrium, are of weights W and W_1 respectively. Determine the weight of the rod and the position of its centre of gravity.

19. A heavy straight rod AB, of length c, balances about a point at a distance a from A when a mass of weight P is suspended from A. It balances about a point distant b from B when a mass of weight Q is suspended from B. Determine the weight of the rod and the position of its centre of gravity.

20. A solid cone of weight W, height h, and radius of base r, rests with its slant side in contact with a smooth horizontal plane. Its centre of gravity divides the straight line joining the vertex to the

centre of the base in the ratio 3 : 1. Find (i) the least vertical force which, applied at the vertex, will move the cone; and (ii) the force at the vertex which will keep it at rest with its axis horizontal.

21. Two light rods OA and OB, of lengths a and b respectively, are rigidly connected at O and inclined at an angle 2α. Masses of weight W and W' are attached at A and B respectively. If the system can turn freely about O, show that, in the position of equilibrium, the bisector of the angle AOB is inclined to the horizontal at an angle

$$\tan^{-1}\left\{\frac{Wa+W'b}{Wa-W'b}\cot\alpha\right\}.$$

22. AOB is a crooked lever, of no appreciable weight, of which O is the fulcrum. When masses of weight P and Q are suspended from A and B respectively, the lever rests with OB horizontal. When Q is changed into Q', the lever rests with OA horizontal. Determine the angle AOB.

23. OBA is a lever of no appreciable weight, bent at B at an angle of 120°, and capable of turning freely about O. If OB and BA are of lengths 3 and 5 inches respectively, find what force applied at A will keep the lever in equilibrium with OB horizontal, when a mass of one hundredweight is suspended from B—

(i) When the applied force is vertical.

(ii) When horizontal.

(iii) When as small as possible.

24. Two levers OA and OB, of inappreciable weight and of lengths 3 and 4 feet respectively, can turn freely in a vertical plane about a common fulcrum O, and their middle points are connected by a fine string whose length is 2½ feet. Find the least force which, applied at A, will keep OB horizontal, when a mass of 12 pounds is suspended from B. Find also the tension of the string.

25. Two fine straight rods AOB and COD, of given lengths, are freely jointed together at O, the lengths BO and DO being given; and the whole is laid on a smooth horizontal table. If A and C are connected by a fine string of given length, and B and D are pulled apart by two forces each of given magnitude P in the straight line BD, show how to determine the tension of the string.

Draw a diagram showing the forces acting on the lever AOB.

26. Two levers AOB and COD, of inappreciable weight, whose lengths are 8 and 9 inches respectively, are freely jointed together at O, 4 inches from B and D. If A and C are connected by a fine light string 3 inches long, and B and D are pulled apart by forces each equal to 10 pounds' weight in the straight line BD, find the tension of the string.

27. A cylindrical column of height $h+k$ and weight W, whose centre of gravity coincides with its centre of figure, stands on a circular base of radius a. A light cord, fastened to the top of the column, reaches to a point on the ground distant $h-k$ from the base. What force must be applied along the cord to upset the column?

28. In the preceding question, show that the column can be upset with the least exertion, if the cord is attached at a point distant

$\sqrt{h^2+k^2}$ above the ground. Show also that the force applied is diminished in the ratio $h^2+k^2 : h^2-k^2$.

29. A cubical block, of edge a, is being turned over by a crowbar of length c, applied at the middle point of the lower edge of one end of the block in a vertical plane through its centre of figure. Find what force must be applied at right angles to the crowbar at one end, so that it may rest inclined at an angle 2α to the horizontal, the lower face of the block being then inclined at an angle α to the horizontal. The block is of given weight W, and its centre of gravity coincides with its centre of figure.

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CHAPTER VII

COPLANAR FORCES

***95. Reduction of any System of Coplanar Forces.**—*Any system of coplanar forces can always be reduced to a single force or to a couple, except when they are in equilibrium.*

We can always find the resultant of two coplanar forces except when they are equal and opposite.

Now consider three forces. They cannot be such that *each* is equal and opposite to *each* of the other two. Thus two of the three forces can be replaced by their resultant, so that three forces can always be reduced to two.

Therefore four forces can always be reduced to three and then to two.

Proceeding in this way we see that any number of coplanar forces can always be reduced to two.

Now two forces are equivalent to a resultant except when they are equal and opposite. (i) If they are equal and opposite and in the same straight line, they are in equilibrium. (ii) If they are equal and opposite and not in the same straight line, they constitute a couple.

Thus any system of coplanar forces can always be reduced to a single force or to a couple, except when they are in equilibrium.

In the following article we give a method for determining the resultant of any system of coplanar forces. It constitutes an independent proof of the present proposition.

96. Resultant of Coplanar Forces.—*To find the resultant, by geometrical construction, of any given system of forces acting in one plane upon a rigid body.*

Let a number of given forces P, Q, R, S, T act in one plane along known lines p, q, r, s, t respectively.

From any suitable point A, and with any suitable scale, draw AB to represent the forces P. From B draw BC to repre-

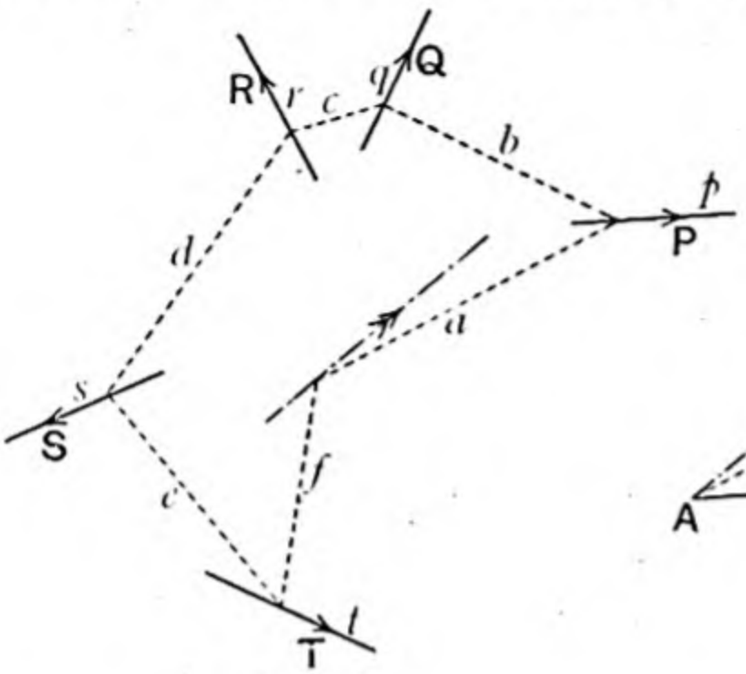


Fig. 97.

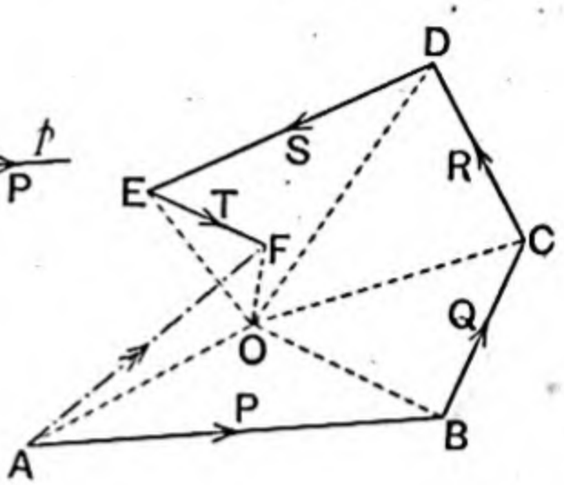


Fig. 97a.

sent the force Q. Similarly, draw CD, DE, EF to represent the forces R, S, T respectively.

We shall prove that the straight line from A, where we started, to F, where we finished, represents the resultant in magnitude and direction, and that its line of action may be determined.

Take any point O in the force diagram, and connect it with the points A, B, C, D, E, F.

From any point in the line p draw straight lines a and b parallel to OA and OB respectively. From the intersection of b and q draw c parallel to OC. From the intersection of c and r draw d parallel to OD. From the intersection of d and s draw e parallel to OE. From the intersection of e and t draw f parallel to OF.

We shall prove that the line of action of the resultant passes through the intersection of a and f.

The force P may be replaced by two forces, represented by AO and OB, acting along the lines a and b respectively; the force Q by two forces, represented by BO and OC, acting along the lines b and c respectively; the force R by two forces, represented by CO and OD, acting along the lines c and d respectively; the force S by two forces, represented by DO and OE, acting along the lines d and e respectively; the force T by two forces, represented by EO and OF, acting along the lines e and f respectively.

Let the given forces be replaced by these pairs of components. Then, removing the pairs of equal and opposite forces which act along the lines b, c, d, e, we see that the whole system is equivalent to two forces, represented by AO and OF, acting along the lines a and f respectively.

Hence the resultant of the system is a force represented by AF, and acts along a straight line drawn parallel to AF through the intersection of a and f.

97. The broken line $ABCDEF$ is called a *Force Polygon*, and is said to *close* when the final point F coincides with the initial point A.

The polygon formed by the lines a, b, c, d, e, f is called a *Funicular Polygon*, and is said to *close* when the lines a and f are in one and the same straight line.

98. In planning out the force diagram, the forces may be taken in any order; the point O may be taken in any position; also, in constructing the funicular polygon, we may start with any point in the line of action of one of the forces. In all cases the same result will be obtained. For the system cannot admit of two resultants, different in magnitude or in direction, or in line of action, since such resultants would have to be equivalent to one another, which is impossible by Art. 14.

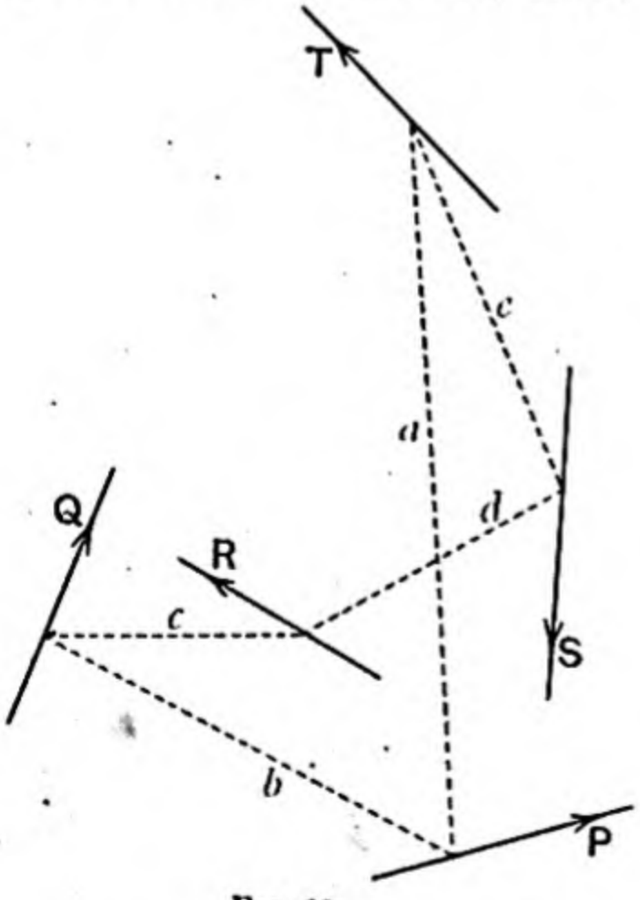


FIG. 98.

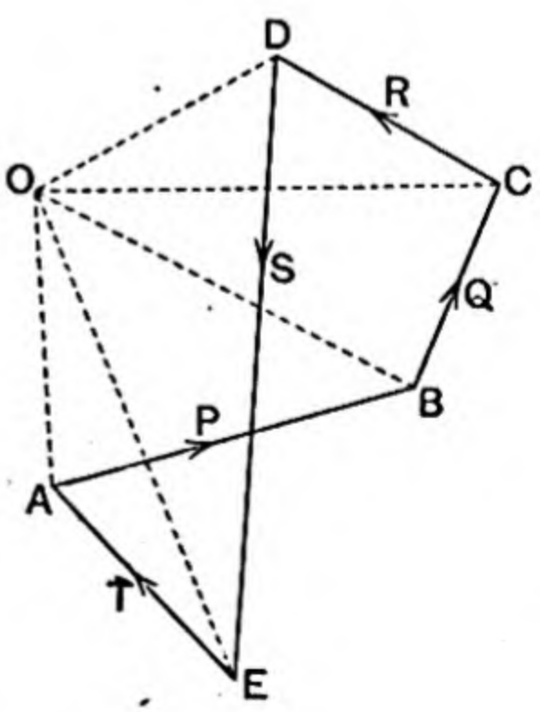


FIG 98a.

99. Geometrical Conditions of Equilibrium. — For the

system to be in equilibrium it is necessary and sufficient that the two forces represented by AO and OF, acting along the lines a and f respectively, should be equal and opposite and act along the same straight line. That is, F must coincide with A, while f and a must be in one and the same straight line.

Thus, for equilibrium, *both polygons must close.*

Figs. 98 and 98*a* are the figures for a system of five forces P, Q, R, S, T in equilibrium.

If the force polygon closes, but not the funicular polygon, the lines a and f become parallel, and in this case the system reduces to a *couple.*

For further information on this subject, the student is referred to the author's *Elementary Geometrical Statics.*

100. Principle of Resolved Parts.—*The sum of the resolved parts in any direction of any number of coplanar forces is equal to the resolved part of their resultant in that direction.*

This is proved in the same way as in Art. 57 by projecting the sides of the force polygon upon any line drawn in the direction in which the forces are to be resolved.

The student should notice the following particular cases :—

(i) The sum of the resolved parts of the forces in the direction of the resultant is equal to the resultant.

(ii) The sum of the resolved parts of the forces in a direction perpendicular to the resultant is zero.

(iii) If the forces are in equilibrium, the force polygon closes, so that the sum of the resolved parts of the forces in every direction is zero.

(iv) If the forces reduce to a couple, the force polygon closes, so that again the sum of the resolved parts of the forces in every direction is zero.

**101.* *If the sum of the resolved parts in a certain direction of any number of coplanar forces is zero, then their resultant, if it exists, is at right angles to that direction.*

This is proved by projecting in the same way as in Art. 58.

**102.* *If the sums of the resolved parts in two directions of any number of coplanar forces are separately zero, then the forces must either be equivalent to a couple or be in equilibrium.*

If the forces were equivalent to a resultant, it would, by the preceding article, be perpendicular to each of the two given directions, which is impossible. Hence the forces cannot reduce

to a resultant, and must, therefore, be equivalent to a couple or be in equilibrium.

103. Principle of Moments.—*The sum of the moments about any point of any number of coplanar forces is equal to the moment of their resultant about that point.*

Referring to the figures of Art. 96, the force P can be replaced by two forces, represented by AO and OB, acting in the lines a and b respectively; the force Q can be replaced by two forces, represented by BO and OC, acting in the lines b and c respectively; and so on.

In each case we know that the moment of the force about any point is equal to the sum of the moments of its components about that point.

Therefore the sum of the moments about any point of the forces P, Q, R, S, T

= moment of force represented by AO acting in the line a
+ " " OB " b
+ " " BO " b
+ " " OC " c
. . . . . .
+ " " EO " e
+ " " OF " f
= moment of force represented by AO acting in the line a
+ " " OF " f
= moment of resultant about the same point.

If the force polygon closes, but not the funicular polygon, the system reduces to a couple, and the sum of the moments of the forces about any point is equal to the moment of the couple, and therefore constant for all positions of the point.

If the force polygon closes, and also the funicular polygon, the system is in equilibrium, and the sum of the moments of the forces, being equal to the sum of the moments of two equal forces which act in opposite directions along the same straight line, vanishes.

The student should notice that

(i) If the system is in equilibrium, the sum of the moments of the forces about every point vanishes.

(ii) If the system has a resultant, the sum of the moments of the forces about every point, situated in the line of action of the resultant, vanishes.

(iii) If the system reduces to a couple, the sum of the

moments of the forces is a constant quantity, not zero, for all positions of the point about which moments are taken.

*Otherwise.—Let the system be reduced to two forces in the manner explained in Art. 95. The process is that of repeatedly choosing two forces which do not form a couple, and replacing them by their resultant, until there are only two forces left.

Now, in every case, the sum of the moments of the two forces is equal to the moment of the resultant which takes their place.

Thus, at every reduction, the sum of the moments of the forces is not altered.

Hence, the sum of the moments about any point of any number of coplanar forces is equal to the sum of the moments about the same point of two forces to which they are equivalent, and therefore to the moment of their resultant about that point, if they have a resultant.

*104. *If the sum of the moments of a system of forces about a certain point is zero, then the forces are not equivalent to a couple, but either are in equilibrium or have a resultant whose line of action passes through that point.*

The forces cannot reduce to a couple, since in that case the sum of their moments would not vanish about any point.

If the forces are equivalent to a resultant, then the moment of that resultant about the point must vanish. This can happen only when its line of action passes through the point.

*105. *If the sum of the moments of a system of forces about each of two given points vanishes, then either the forces are in equilibrium, or they are equivalent to a resultant acting along the line joining the two given points.*

This follows at once from the preceding article.

*106. **Analytical Conditions of Equilibrium.**—*If the sum of the resolved parts in each of* TWO *directions, and the sum of the moments about* ONE *point, of any system of coplanar forces, vanish separately, then the system is in equilibrium.*

The first two conditions ensures that the forces do not reduce to a resultant; while the third condition ensures that they do not reduce to a couple; hence they must be in equilibrium.

*107. *If the sums of the moments of any system of coplanar forces about* THREE *points in their plane (not situated in the same straight line) vanish separately, then the system is in equilibrium.*

For, if the system were not in equilibrium, it would by Art.

104 be equivalent to a single resultant whose line of action passes through each of the three points, which is impossible.

*108. *If the sum of the moments about each of* TWO *points, and the sum of the resolved parts in* ONE *direction (not perpendicular to the straight line joining the two points) of any system of coplanar forces vanish separately, then the system is in equilibrium.*

The first two conditions ensure that if the system were not in equilibrium it would, by Art. 105, be equivalent to a resultant acting along the line joining the two points, while the third condition ensures that such resultant would have to be perpendicular to the one direction, and these two results are inconsistent. Thus the system must be in equilibrium.

*109. We see then that three conditions are necessary and sufficient for the equilibrium of a system of coplanar forces. In dealing with the equilibrium of any such system, we can write down three independent equations. We either resolve twice and take moments once, or resolve once and take moments twice, or take moments three times.

In practice, the first of these methods is generally found the most convenient, as the magnitude and direction only of a force enters into consideration in the process of resolving, while the perpendicular distance of its line of action from a point is required in the process of taking moments. Also, in resolving twice, it is in general found convenient to choose the directions at right angles.

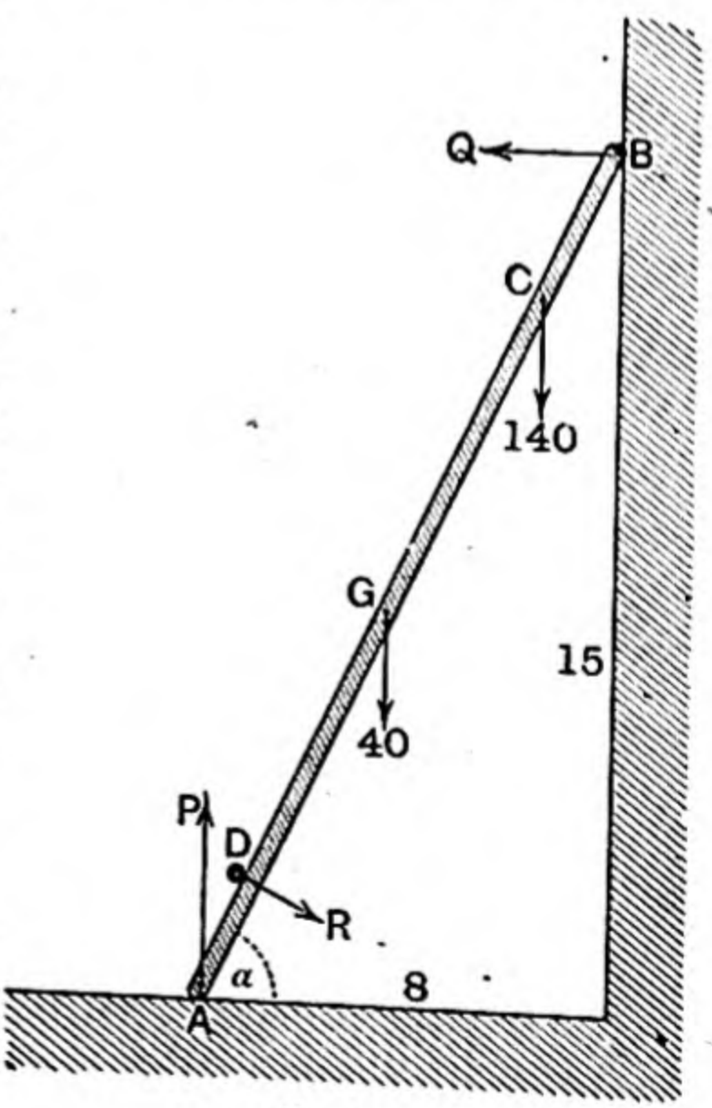


FIG. 99.

Example.—Fig. 99 represents a ladder resting against a smooth wall at *B* and the smooth ground at *A*. The bottom of the ladder is 8 feet from the wall and the top 15 feet from the ground. A man of 10 stone stands on the ladder at a point C distant 3 feet from the top, and the ladder is prevented from

slipping by a smooth rail D, fixed at a point 2 feet from A. The ladder itself weighs 40 pounds, and its centre of gravity G is 8 feet from A. It is required to determine the pressures at A, B, and D.

The length of the ladder is $\sqrt{8^2+15^2}$ feet $=17$ feet.

The pressures upon the ladder at A, B, and D are vertical, horizontal, and perpendicular to AB respectively. Let their magnitudes be P, Q, and R pounds' weight respectively, and let α be the inclination of the ladder to the horizontal.

Considering the equilibrium of the ladder and the man as one system, we have—

$$\begin{cases} \text{Resolving horizontally,} \\ \qquad R\sin\alpha = Q. \\ \text{Resolving vertically,} \\ \qquad P = 180 + R\cos\alpha. \\ \text{Taking moments about } A, \\ \qquad 2R + (40\times 8\cos\alpha) + (140\times 14\cos\alpha) = 15Q. \end{cases}$$

These three equations are equivalent to

$$\begin{cases} \frac{15}{17}R = Q \\ P = 180 + \frac{8}{17}R \\ 2R + \left(2280\times\frac{8}{17}\right) = 15Q. \end{cases}$$

Eliminating Q between the first and third of these equations, we have—

$$34R + 18240 = 225R$$

$$\therefore R = \frac{18240}{191} = 95{\cdot}5\ldots.$$

And $$Q = \frac{15}{17}R = 84{\cdot}3\ldots.$$

Also $$P = 180 + \frac{8}{17}R = 224{\cdot}9\ldots.$$

Thus the pressures upon the ground, the wall, and the rail are $224{\cdot}9\ldots$, $84{\cdot}3\ldots$, and $95{\cdot}5\ldots$ pounds' weight respectively.

*** 110. Equilibrium of a Body with one Point fixed.**—Suppose that a rigid body is capable of turning freely about a fixed point, and is acted upon by a system of forces, in addition to the force of constraint, all situated in a plane passing through the fixed point. Then, for equilibrium, it is necessary and sufficient that the sum of the moments of the forces about the fixed point should be zero. For, if this condition is satisfied, either the forces are in equilibrium among themselves, or they have a resultant whose line of action passes through the fixed point; and, in the latter case, the force of constraint will be equal and opposite to that resultant, and such as to balance the other forces.

If we assume a magnitude and direction for the force of constraint, R, and include it among the other forces, we may apply the conditions of equilibrium of Art. 106. Thus, taking moments about the fixed point, the force R does not appear in our equation, and we have the necessary and sufficient relation between the other forces; and, resolving along two directions (preferably at right angles), we have two equations to determine the magnitude and direction of the force R.

Example.—Fig. 100 represents a uniform straight rod AFB, of length 1 foot and weighing 8 pounds, capable of turning freely in a vertical plane about the fixed point F, situated 4 inches from B. A mass of 10 pounds is suspended from B, and the rod is supported in a horizontal position by a fine light string AC, inclined at an angle of 30° to the horizontal. It is required to determine the tension of the string and the force of constraint at F.

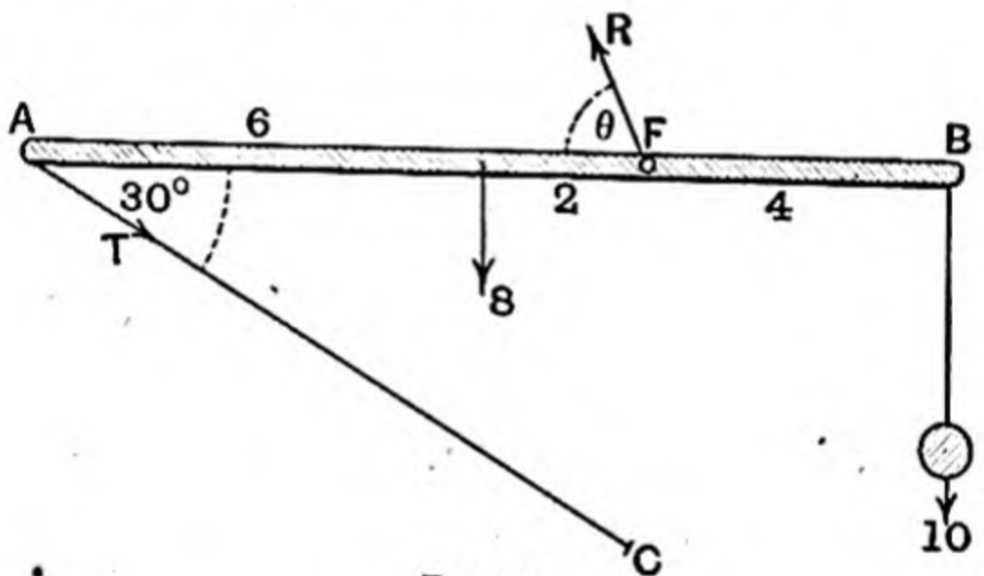


FIG. 100.

Let T pounds' weight be the tension of the string. Then, taking moments about F, we have—

$$T \cdot 8 \sin 30° + (8 \times 2) = (10 \times 4),$$

whence

$$T = 6.$$

Suppose that the action of the hinge upon the rod at F is R pounds' weight, in a direction making an angle θ with FA. Then, resolving vertically and horizontally, we have—

$$\begin{cases} R \sin\theta = 18 + T \sin 30° = 21, \\ \text{and } R \cos\theta = T \cos 30° = 3\sqrt{3}, \end{cases}$$

$$\therefore R^2 = 21^2 + 27 = 468,$$

so that

$$R = 21{\cdot}63\ldots,$$

and

$$\tan\theta = \frac{7}{3}\sqrt{3} = 4{\cdot}04145\ldots.$$

Thus the tension of the string is 6 pounds' weight, and the force of constraint at F is 21·63... pounds' weight in a direction making an angle $\tan^{-1} 4{\cdot}04145\ldots$ with the horizontal.

*EXAMPLES XIV

1. A uniform ladder, of weight w, rests with one end against a smooth vertical wall, to which it is inclined at an angle α, and with the other end upon the smooth horizontal ground. A man, of weight W, stands upon the ladder at a point three-quarters of the way up, and it is kept from slipping by means of a smooth peg at its lowest point. Determine the pressures upon the peg, the ground, and the wall.

2. A ladder AB, of weight w and length $5a$, having its centre of gravity at a distance $2a$ from the lower end A, rests at B against a smooth vertical wall BC, and is prevented from slipping over the smooth horizontal ground by means of a fine light horizontal string AC of length $3a$. Find the tension of the string when a man of weight W stands (i) halfway up the ladder, (ii) at the top of the ladder.

3. A uniform ladder ADB, of weight w and length $6a$, rests at B against a smooth vertical wall BC, and is prevented from slipping over the smooth horizontal ground by means of a fine light horizontal string DC, of length $3a$, attached to the ladder at a point D, one-sixth of the way up, and to the wall at a point C situated vertically below B. Show that, as a man of weight W ascends the ladder, the tension of the string gradually increases from $\frac{9}{20}w$ to $\frac{9}{20}(w+2W)$.

4. A uniform rod AOB, of length one foot and weighing 2 pounds, is capable of turning freely about a point O, distant 3 inches from A. It is supported in a horizontal position, with a mass of 4 pounds suspended from B, by means of a fine light string, 5 inches long, connecting A with a point C, fixed vertically below O. Find the tension of the string and the reaction at O.

5. A uniform rod AOB, of weight W, is capable of turning freely about a fixed point O, and rests with its lower end B against a smooth vertical wall, to which it is inclined at an angle $\tan^{-1}4$. If O divides AB in the ratio $1:2$, and if a mass of weight $\frac{7}{8}W$ is suspended from A, determine the pressure on the wall and the reaction at O.

6. A uniform rod ACB, of length $2a$ and weight W, rests against a smooth horizontal plane at A and against a smooth fixed rail at C. It is prevented from slipping by means of a fine light string, connecting A with a point D situated in the horizontal plane vertically below C. If AC is of length c, and if the rod rests inclined at an angle α to the horizontal, determine the tension of the string, and show that A will not remain in contact with the horizontal plane unless

$$c > a\cos^2\alpha.$$

7. A uniform rod ACB, of length $2a$ and weight W, rests with its lower end A upon a smooth horizontal plane, and at C against a smooth rail fixed at a height h above the plane. It is prevented from slipping by means of a fine light string, of length l, connecting A with a point D situated in the horizontal plane vertically below C. Determine the tension of the string and the pressure on the peg.

8. The lower extremity A of a uniform beam AB, of length c and weight W, rests on the ground at the foot of a vertical wall AC, its upper extremity B being attached by a fine light string of length a

to a point C situated in the wall at a distance b vertically above A. Determine the tension of the string and the pressure of the beam upon the ground at A.

9. A square board $ABCD$ is placed upon a smooth horizontal table, and a given force P acts from E, the middle point of AB, towards F, the middle point of CD. Determine the magnitudes of three forces X, Y, Z, which, acting along BC, CA, AB respectively, will make equilibrium with P.

10. A uniform square lamina $ABCD$, of weight W, is supported with BC vertically downwards by three forces as follows:—P acting along BC, Q acting along DB, R acting along CE, where E is the middle point of AB. Determine the values of P, Q, R.

11. A uniform square lamina $ABCD$, of weight W, is constrained at A and D to remain in contact with a smooth fixed vertical bar, D being below A, and rests with the corner C upon a smooth plane inclined at an angle α to the horizontal. Determine the reactions at A, D, C.

12. A triangular lamina ABC, of no appreciable weight, whose sides BC, CA, AB are respectively 18, 24, and 30 inches in length, is placed in a vertical plane with BC and CA resting respectively upon two fixed smooth pegs D and E, situated 20 inches apart in the same horizontal line. If masses, each of weight W, are suspended from A and B, and the triangle is kept with AB horizontal by means of a fine light string connecting C with the peg D, find the tension of the string and the pressures on the pegs.

13. The sides AB, BC, CD, DA of a trapezium are of lengths c, a, b, d respectively, AD being parallel to BC. Find what forces must act along BA, DC, DA, so as to be in equilibrium with a given force P acting along BC.

14. $ABCDEF$ is a regular hexagon. Find what forces must act along CE, CB, FA, so as to be in equilibrium with a given force P acting along AE.

15. A straight bar AGB, of weight W, having its centre of gravity G at distances a and b from A and B respectively, is supported in a horizontal position by means of two fine light strings AC and BD, so that the angles CAB and ABD are 120° and 150° respectively. Find what force must be applied at E, the middle point of AG, in direction EF, so as to preserve equilibrium, the angle AEF being 60° and F below AB. Determine also the tensions of the strings AC and BD.

16. A straight uniform bar $AEFB$, of length 1 foot and weight W, is supported in a horizontal position by means of two fine light strings AC and BD, so that the angles CAB and ABD are 120° and 150° respectively. A mass of weight $2W$ is suspended from E, at a distance of 1 inch from A. Find the weight of a mass which must be suspended from F, distant 4 inches from A, in order to preserve equilibrium. Determine also the tensions of the strings AC and BD.

17. ABB_1A_1 is a fine straight bar of no appreciable weight, the portions AB, BB_1, B_1A_1 being of lengths a, b, a_1 respectively. A mass of weight W is suspended from A, and a mass of unknown weight is suspended from A_1. The rod is supported in a horizontal position by means of two forces applied at B and B_1, in directions BC and B_1C_1 respectively, so that the angles ABC and $A_1B_1C_1$ are each of magnitude

a. Determine the weight of the mass suspended from A_1, and the magnitudes of the forces applied at B and B_1.

18. A uniform beam AB, of weight W, is supported by two fine light strings AC and BD, the latter being vertical and the angles CAB and ABD being each $\pi - \alpha$. What horizontal force must be applied at B in order to preserve equilibrium? Find also the tensions of AC and BD.

19. A fine heavy bar, of weight W, whose centre of gravity divides it into two portions in the ratio $a : b$, rests on two smooth inclined planes whose intersection is a horizontal line, the rod lying in a vertical plane perpendicular to their line of intersection. If the planes are inclined at angles α and β respectively to the horizontal, determine the inclination of the rod to the horizontal in the position of equilibrium, and find the pressures on the planes.

20. A uniform rectangular lamina $ABCD$, of weight W, rests upon a smooth horizontal plane EBF and between two smooth vertical walls AE and CF, the plane of the lamina being perpendicular to each wall. If AB is inclined to the horizontal at an angle θ, and if AB and BC are of lengths a and b respectively, determine the pressures at A, B, C.

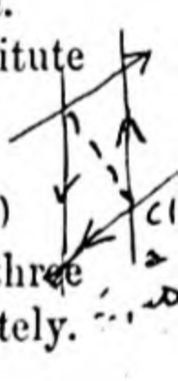
21. A straight ladder AB is inclined at an angle of 60° to the horizontal, and rests upon a smooth horizontal plane AC and against a smooth wall BC inclined at an angle of 75° to the horizontal. The ladder is prevented from slipping by means of a fine string connecting A with C. If the tension of the string is one-fifth of the weight of the ladder, determine the position of the centre of gravity of the ladder.

***111. Couples.**—*Two couples, of equal and opposite moment, acting in the same plane upon a rigid body, are in equilibrium.*

The sum of the moments of the four forces, which constitute the two couples, about *every* point

= the sum of the moments of the two couples

= zero.



Thus the sums of the moments of the four forces about three points (not situated in the same straight line) vanish separately.

Therefore, by Art. 107, the system is in equilibrium.

From this proposition it follows that:

Any couple may be replaced by any other couple whose moment is of the same magnitude and sense.

For each couple could in turn be balanced by the same couple, namely, one of equal and opposite moment.

***112.** *Any number of couples, acting in the same plane upon a rigid body, whose moments are such that their algebraic sum is zero, form a system in equilibrium.*

The sum of the moments of the forces, which constitute the couples about *every* point

= sum of moments of the couples

= zero.

Therefore, by Art. 107, the system is in equilibrium.

From this proposition it follows that:

Any number of couples, acting in the same plane upon a rigid body, are equivalent to a single couple whose moment is equal to the algebraic sum of the moments of the couples.

For such a couple could be balanced by the same couple which would balance all the others.

***113.** *To find the resultant of a given force and a given couple.*

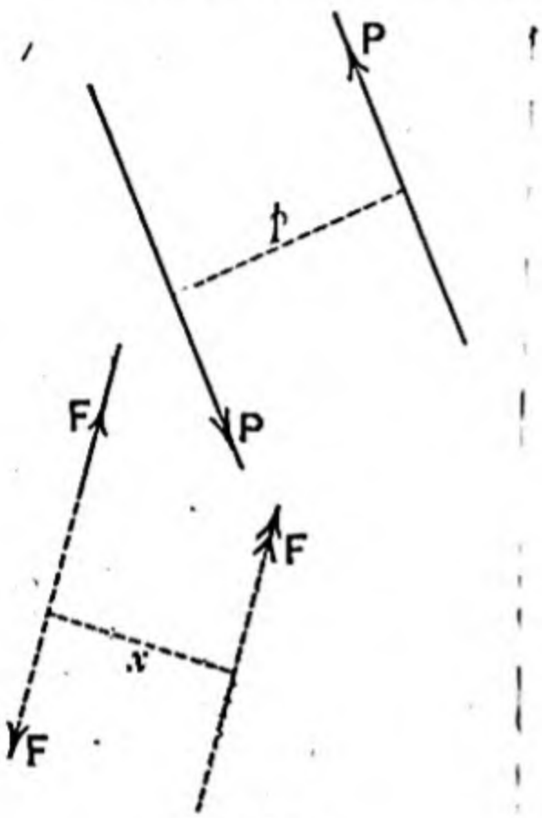


FIG. 101.

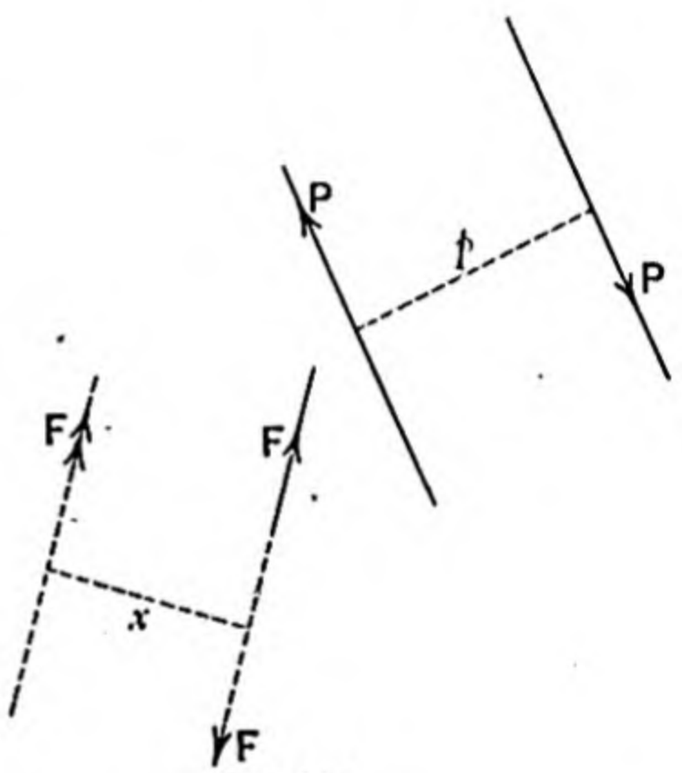


FIG. 102.

Let F be the given force, and P,P the given couple acting at an arm p.

The couple P,P can be replaced by a couple F,F, acting at an arm x where $Fx = Pp$, and the couple F,F can be arranged so that one of the forces F acts along the same straight line as the given force F, but in the opposite direction. Thus the resultant is the remaining force F of the couple so altered and arranged.

Hence the resultant of the system is a force F, acting along a line parallel to the given force F, at a distance $\frac{P \cdot p}{F}$ from its line of action.

114. Analytical Determination of Resultant.—*To find by calculation the resultant of a given system of coplanar forces.*

Let P be one of a given system of forces acting upon a rigid body in one plane. Let O be a fixed point, and OX a fixed straight line, in the plane. Let the force P act at an angle α

with OX along a line which meets OX at a distance a from O.

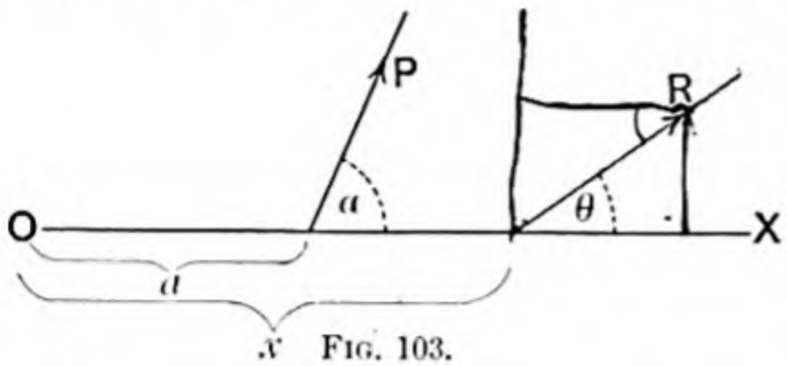


FIG. 103.

Suppose that the system is equivalent to a resultant of magnitude R acting at an angle θ with OX along a line which meets OX at a distance x from O.

Then, resolving along and perpendicular to OX, we have

$$\begin{cases} R\cos\theta = \Sigma(P\cos\alpha) \\ R\sin\theta = \Sigma(P\sin\alpha). \end{cases}$$

These two equations determine the magnitude and direction of the resultant, as in Art. 60.

Again, taking moments about O, we have

$$R\,.\,x\sin\theta = \Sigma(P\,.\,a\sin\alpha)$$

$$\therefore\ x = \frac{\Sigma(Pa\sin\alpha)}{R\sin\theta} = \frac{\Sigma(Pa\sin\alpha)}{\Sigma(P\sin\alpha)}.$$

This determines the position of the line of action of R.

If $\Sigma(P\cos\alpha)$ and $\Sigma(P\sin\alpha)$ both vanish, R vanishes; and if, in addition, $\Sigma(Pa\sin\alpha)$ does not vanish, then x is infinite. Thus the system appears to reduce to an infinitely small resultant acting along a line at an infinite distance from O. We know, by Arts. 102 and 103, that in this case the system is equivalent to a couple of moment $\Sigma(Pa\sin\alpha)$.

Example.—Suppose that R is the resultant of the forces 2, 3, 4, 5 indicated in Fig. 104.

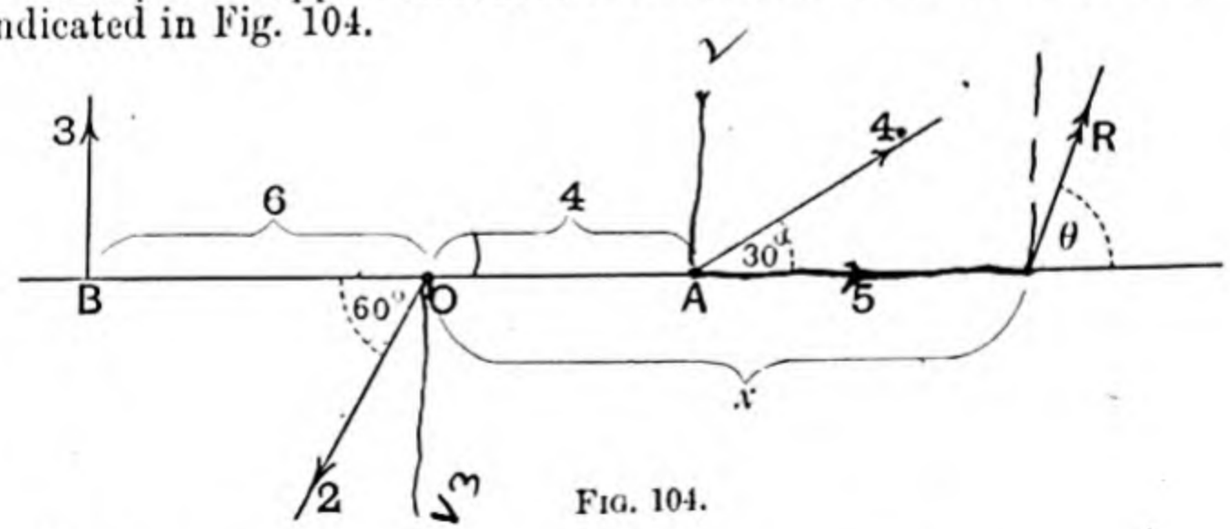


FIG. 104.

Then

$$\begin{cases} R\cos\theta = 5 + 4\cos 30^\circ - 2\cos 60^\circ \\ \qquad\quad = 4 + 2\sqrt{3} \\ R\sin\theta = 4\sin 30^\circ + 3 - 2\sin 60^\circ \\ \qquad\quad = 5 - \sqrt{3} \end{cases}$$

$$\therefore\ R^2=(4+2\sqrt{3})^2+(5-\sqrt{3})^2=56+6\sqrt{3}=66{\cdot}3923\ldots$$
$$\therefore\ R=8{\cdot}148\ldots$$
$$\text{and } \tan\theta=\frac{5-\sqrt{3}}{4+2\sqrt{3}}=\frac{(5-\sqrt{3})(2-\sqrt{3})}{2}=\frac{13-7\sqrt{3}}{2}={\cdot}4378\ldots$$

Also, taking moments about O,

$$\mathrm{R}\,.\,x\sin\theta=(4\times 4\sin 30^\circ)-(3\times 6)=-10$$
$$\therefore\ x=-\frac{10}{R\sin\theta}=-\frac{10}{5-\sqrt{3}}=-\frac{10(5+\sqrt{3})}{22}=-3{\cdot}06\ldots$$

Thus the resultant is of magnitude $8{\cdot}148\ldots$ acting at an angle $\tan^{-1}{\cdot}4378\ldots$ with BA along a line which meets OB at a point $3{\cdot}06\ldots$ inches from O.

*** 115.** Ex. 24.—*A beam AOB, of weight W, is capable of turning freely about the fixed point O. It is sustained in a position inclined to the vertical by a fine string, which is attached to the lower end B and passes over a small smooth pulley C, situated vertically above O, supporting at its other extremity a mass of weight w. It is required to determine the inclination of the beam to the vertical in the position of equilibrium, and to find the strain upon the hinge.*

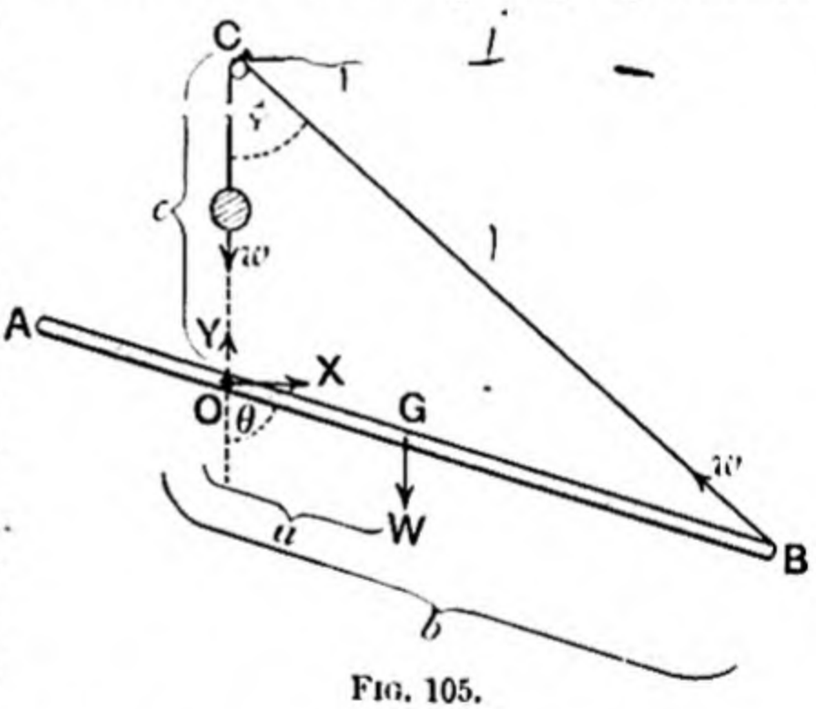


FIG. 105.

Let G be the centre of gravity of the beam, and let OG, OB, OC be of lengths a, b, c respectively. Let θ and ϕ be the inclinations of the rod and the string to the vertical in the position of equilibrium.

The tension of the string is of magnitude w throughout. Taking moments about O for the equilibrium of the beam, we have

$$Wa\sin\theta=wc\sin\phi\quad .\quad .\quad .\quad \text{(i)}.$$

Also, from the geometry of the figure,

$$\frac{c}{b}=\frac{\sin OBC}{\sin OCB}=\frac{\sin(\theta-\phi)}{\sin\phi},$$
$$\therefore\ c\sin\phi=b\sin(\theta-\phi)$$

whence,
$$b\sin\theta\cos\phi=(b\cos\theta+c)\sin\phi\quad .\quad .\quad \text{(ii)}.$$

From the equations (i) and (ii) we have

$$\begin{cases} wbc \sin\phi = Wab \sin\theta \\ wbc \cos\phi = (b\cos\theta + c)\, Wa \end{cases}$$

$\therefore$ eliminating ϕ,

$$w^2b^2c^2 = W^2a^2(b^2+c^2) + 2W^2a^2bc\cos\theta,$$

$$\therefore \cos\theta = \frac{w^2b^2c^2 - W^2a^2(b^2+c^2)}{2W^2a^2bc}.$$

This determines the inclination of the beam.

Also,

$$\cos\phi = \frac{Wa}{wbc}(b\cos\theta + c)$$

$$= \frac{w^2b^2c^2 - W^2a^2(b^2-c^2)}{2Wwab^2c^2}.$$

Let X and Y be the horizontal and vertical components of the action of the hinge upon the beam, as marked in the figure. Then, resolving horizontally and vertically, we have

$$\begin{cases} X = w\sin\phi \\ Y = W - w\cos\phi. \end{cases}$$

Thus, the strain upon the hinge

$$= \sqrt{X^2+Y^2}$$

$$= \sqrt{W^2+w^2-2Ww\cos\phi}$$

$$= \sqrt{W^2+w^2-\frac{w^2b^2c^2 - W^2a^2(b^2-c^2)}{abc^2}}$$

$$= \sqrt{W^2\left(1+\frac{ab}{c^2}-\frac{a}{b}\right) - w^2\left(\frac{b}{a}-1\right)}.$$

* EXAMPLES XV

1. $ABCD$ is a parallelogram; three forces are represented in magnitude, direction, and position by AB, BC, DC. Determine their resultant completely.

2. Four forces act along, and are represented by, the sides AB, BC, CD, AD of a parallelogram $ABCD$. Find their resultant and its line of action.

3. Four forces act along, and are represented by, the sides AB, BC, DC, AD of any quadrilateral $ABCD$. Find their resultant and its line of action.

4. Four forces act along, and are represented by, the sides AB, CB, CD, AD of any quadrilateral $ABCD$. Find their resultant and its line of action.

5. Prove the following construction for determining the resultant of four given forces P, Q, R, S acting along four given lines p, q, r, s respectively.

Take AB, BC, CD, DE to represent P, Q, R, S respectively. Let p and q intersect in H, while r and s intersect in K. Draw HL parallel to AC and KL parallel to CE, meeting in L. Then the resultant is represented by AE and acts along a line through L.

6. In the preceding example, show that, for the system to be in equilibrium, it is necessary and sufficient that E should coincide with A and that HK should be parallel to AC.

7. Four coplanar forces P, X, Y, Z, in equilibrium, act along four given lines p, q, r, s respectively. If the force P is known, prove the following construction for determining the values of X, Y, Z.

Let p and q intersect in H, while r and s intersect in K. Take AB to represent P and draw BC parallel to q, and AC parallel to HK, meeting in C. Draw CD parallel to r and AD parallel to s, meeting in D. Then BC, CD, DA represent X, Y, Z respectively.

8. In the preceding example, prove that BD is parallel to the line joining the intersection of p and s to the intersection of q and r.

9. Forces P, $P+Q$, $P-Q$ act along the sides BC, CA, AB respectively of an equilateral triangle ABC. Determine their resultant.

10. Forces of 13, 10, 5 pounds' weight act along the sides BC, CA, AB respectively of an equilateral triangle ABC. Find the magnitude and direction of the resultant, and the point where its line of action intersects BC or BC produced.

11. The sides BC, CA, AB of the triangle ABC are of lengths 6, 9, 8 inches respectively. Determine the resultant of forces 8, 12, $2\frac{2}{3}$ pounds' weight acting along BC, CA, AB respectively.

12. Forces of 1, 2, 4, 4 pounds' weight act along the sides AB, BC, CD, DA respectively of a square $ABCD$. Find the magnitude and direction, and the point of application in the line BC, of the force which would balance the system.

13. Forces P, Q, P, Q act along the sides AB, BC, CD, DE respectively of a regular hexagon $ABCDEF$. Determine their resultant.

14. $ABCD$ is a quadrilateral, the angles A and D being right angles and the sides CD, DA, AB of lengths 9, 12, 14 inches respectively. Points E and F are taken in AB at distances of 4 and 9 inches from A respectively, and G is taken in AD at a distance of 3 inches from A. Determine the resultant of the following system of four forces: 9 pounds' weight acting along AD, 13 pounds' weight along BC, 7 pounds' weight along GE, 21 pounds' weight along DF.

15. Three forces act along the sides of a given triangle taken one way round. Prove that if they have any values X, Y, Z subject to the condition that $lX+mY+nZ=0$, where l, m, n are three given numbers, then the line of action of the resultant passes through a fixed point.

In particular, if $X \pm Y \pm Z=0$, prove that the line of action of the resultant passes through the centre of one of the four circles which touch the sides of the triangle.

16. Three forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC. Prove that, if

$$P\cos A+Q\cos B+R\cos C=0,$$

then the line of action of the resultant passes through the centre of the circle which circumscribes the triangle.

17. Three forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC. Prove that, if

$$P \sec A + Q \sec B + R \sec C = 0,$$

then the line of action of the resultant passes through the orthocentre of the triangle.

18. P is the orthocentre of a triangle ABC. Forces proportional to $\sin(A+\theta)$, $\sin(B+\theta)$, $\sin(C+\theta)$ act along AP, BP, CP respectively. Prove that their resultant passes through the centre of the circumscribed circle. What is the value of θ when the forces are in equilibrium?

19. If M_1 and M_2 are the algebraical sums of the moments of a system of coplanar forces about two points A and B, prove that the algebraical sum about a point C lying between A and B will be

$$\frac{M_1 . BC + M_2 . AC}{AB};$$

and *hence* show that, if the algebraical sums of the moments of such a system with respect to three points not in a straight line, are separately equal to zero, then the sum will be zero for any other point in the same plane.

20. A uniform rod AB, of weight W, is capable of turning freely in a vertical plane about a smooth hinge A. It is supported by a fine string BC, which passes over a small smooth pulley C, situated vertically above A, and supports at its other extremity a mass of weight w. If $AC = AB$, find the inclination of the rod to the vertical and the action of the hinge A.

21. A uniform beam AB, of length 4 feet and mass 30 pounds, is capable of turning freely about a fixed point O, situated in the beam at a distance of one foot from A. It is sustained in a position inclined to the vertical by a fine string, which is attached to the lower end B and passes over a small pulley C, situated two feet vertically above O, supporting at its other extremity a mass of 20 pounds. Prove that in the position of equilibrium, the beam is inclined at an angle $\cos^{-1}\frac{1}{4}$ with the vertical, and determine the action at O.

22. A rigid framework, of no appreciable weight, in the form of a triangle ABC, is capable of turning freely about a fixed point A. To the point B a mass of weight W is attached, and AB is pulled aside from the vertical by a horizontal force F applied at C. Show that, in the position of equilibrium, AB is inclined to the vertical at an angle

$$\tan^{-1}\left\{\frac{Fb\cos A}{Fb\sin A + Wc}\right\}.$$

23. The lower end B of a uniform rod AB, of length $2a$ and weight W, rests on a smooth horizontal plane BD, and the upper end A is supported by a fine string AC of length b, the extremity C being fixed at a point whose height above the plane is $2a$. Find (i) the horizontal force which, applied at the lower end of the rod, will cause it to rest at a distance b from the point in the plane immediately below C, (ii) the reaction of the plane at B, and (iii) the tension of the string AC.

24. A straight rod AB, of inappreciable weight, is placed with its

middle point C at the highest point of a fixed horizontal circular cylinder, the rod being at right angles to the axis. If the cylinder is so rough that the rod can tip up without slipping, and if masses of weight W and W' are suspended from A and B respectively, prove that the rod in its new position of equilibrium will be inclined to the horizontal at an angle

$$\frac{l}{a} \cdot \frac{W \sim W'}{W + W'},$$

where $2l$ is the length of the rod and a the radius of the cylinder.

25. Three forces P_1, P_2, P_3, act at the middle points of the sides a, b, c of a triangle ABC, and at right angles to those sides. Show that if they form a system in equilibrium with Q_1, Q_2, Q_3 acting along the same sides, then (the forces being taken in the proper directions)

$$Q_1 \cdot 4\Delta = (P_2 a - P_1 b) b - (P_3 a - P_1 c) c$$

and two similar equations, Δ being the area of the triangle.

26. Four forces in equilibrium act along the sides of a quadrilateral $ABCD$. From a point A_1 in BD a straight line A_1B_1 is drawn parallel to AB to meet AC in B_1; B_1C_1 is drawn parallel to BC to meet BD in C_1; C_1D_1 is drawn parallel to CD to meet AC in D_1. Prove that D_1A_1 is parallel to DA, and that the forces in the lines AB, BC, CD, DA are respectively proportional to A_1B_1, B_1C_1, C_1D_1, D_1A_1.

27. Four forces in equilibrium act along the sides of a quadrilateral $ABCD$. The points A_1, B_1, C_1, D_1 are respectively the centres of the circles which circumscribe the triangles DAB, ABC, BCD, CDA. Prove that the forces in the lines AB, BC, CD, DA are respectively proportional to A_1B_1, B_1C_1, C_1D_1, D_1A_1.

28. Four forces in equilibrium act along lines which bisect at right angles the sides of a quadrilateral. Show that they act all outwards or all inwards, and are proportional to the sides to which they are respectively perpendicular.

29. Forces proportional to the sides of a closed polygon act along lines which bisect at right angles those sides respectively. If, in going round the polygon, the forces act all towards the right or all towards the left, show that they form a system in equilibrium.

30. A given force P, acting along a given line AB, is in equilibrium with three forces X, Y, Z, acting along three given lines A_1B_1, AA_1, BB_1 respectively, the straight lines AB, A_1B_1, AA_1, BB_1 being of lengths c, c_1, a, b respectively. Determine the values of X, Y, Z in each of the following cases:—

(i) A_1B_1 and AB are parallel and in the same direction.
(ii) A_1B_1 and AB are parallel and in opposite directions.
(iii) AB_1 and BA_1 are parallel and in the same direction.
(iv) AB_1 and BA_1 are parallel and in opposite directions.
(v) A, B, A_1, B_1, are points taken one way round on the circumference of a circle.
(vi) A, B, B_1, A_1 are points taken one way round on the circumference of a circle.
(vii) A_1B_1 is perpendicular to AB and towards AB, and BB_1 is perpendicular to AA_1.
(viii) A_1B_1 produced bisects AB and BB_1 produced bisects AA_1.

***116. Systems of Jointed Rods.**—The *analytical* treatment of the equilibrium of systems of jointed rods is exhibited in some of the examples placed at the end of the present chapter. The *geometrical* method of treatment is also exemplified, but requires some preliminary explanation.

***117.** Fig. 106 represents two rods AB and BC of a framework, smoothly jointed together at B by means of a small smooth pin of no appreciable weight or size. The rods and the pin are drawn separated, so as to indicate clearly the forces acting upon the separate pieces.

Let each rod be acted upon by forces in addition to the constraints at its extremities, and let a further force F act upon the pin B.

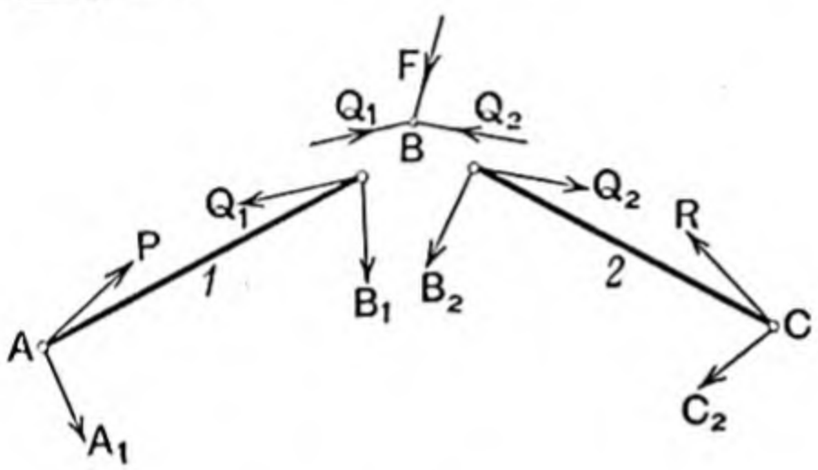


FIG. 106.

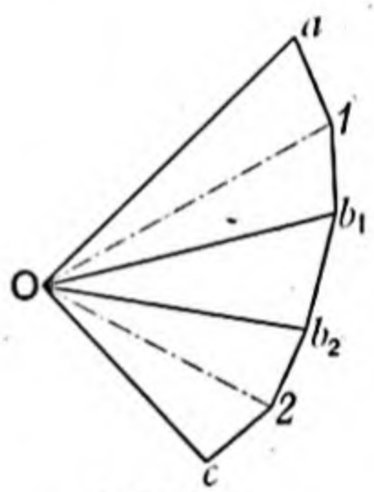


FIG. 106a.

I Let the rods AB and BC be denoted by the figures *1* and *2* respectively. Let the forces that act upon the rod AB (other than the constraints at its extremities) be resolved into two forces A_1 and B_1, acting at A and B respectively. Similarly, let the forces applied to the rod BC be resolved in B_2 and C_2 acting at B and C respectively.

Let P be the reaction at A upon the rod *1*, Q_1 the action between the pin B and the rod *1*, Q_2 the action between the pin B and the rod *2*, R the reaction at C upon the rod *2*.

II (a) Considering the equilibrium of the rod AB, we see that the forces P and A_1 at A balance the forces B_1 and Q_1 at B. Hence the resultant of the first pair must be equal and opposite to the resultant of the second pair, and must be in the same straight line AB.

Thus the force diagram for the rod *1* is the figure $Oa1b_1O$, in which Oa, $a1$, $1b_1$, b_1O represent P, A_1, B_1, Q_1 respectively, and in which $O1$ must be parallel to the rod *1*.

(b) The force diagram for the pin B is Ob_1b_2O, in which b_1b_2 represents F and b_2O represents Q_2.

(c) The force diagram for the rod 2 is $Ob_2 2cO$, in which $b_2 2$, $2c$, cO represent B_2, C_2, R respectively, and in which $O2$ must be parallel to the rod 2.

III (1) The student should notice that if we suppose the forces B_1, F, B_2 to be collected together and applied at the point B, then the line 12 represents their resultant. If we do not require to determine the exact nature of the constraints in the neighbourhood of the joint, we draw the straight line 12 at once, without indicating the points b_1 and b_2.

(2) The straight lines ab_1 and b_2c represent the resultant forces applied to the rods AB and BC respectively, excluding the constraints at their extremities.

(3) If there is no separate force applied to the pin B, then the points b_1 and b_2 coincide, and may be denoted by a single letter b. The action at the joint is then given by Ob.

***118.** Ex. 25.—*Fig.* 107 *represents a framework of seven equal rods, of no appreciable weight, freely jointed together at their extremities so as to form three equilateral triangles, and resting in a vertical plane on horizontal supports at A and C, so that ABC is horizontal. It is required to determine the stresses in the rods AB, BE, ED when the framework is loaded at the joints E and D with masses of weights W_1 and W_2 respectively.*

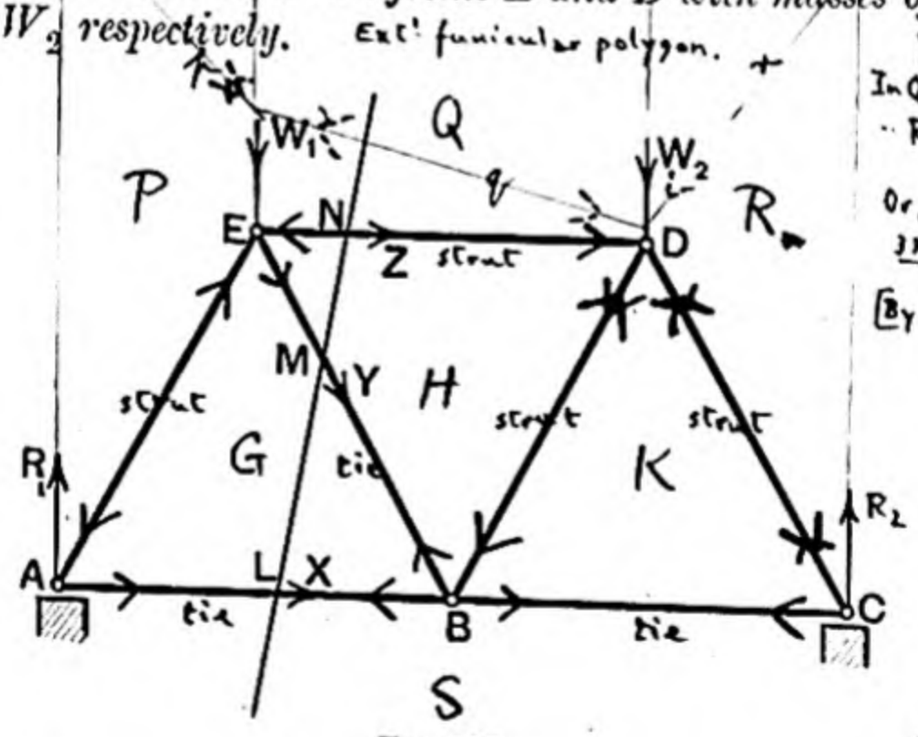


FIG. 107.

Each rod is in equilibrium under forces acting only at its extremities, so that each rod is in a state of direct compression or tension.

(a) Let R be the reaction of the support at A. Then, taking moments about C for the equilibrium of the whole framework, we have

$$4R = 3W_1 + W_2.$$

(b) Now draw the straight line LMN intersecting the rods AB, BE, ED in L, M, N respectively, and consider the equilibrium of the portion $ALMNE$. The actions at L, M, N upon this portion are along the rods intersected. Let these actions be of magnitudes X, Y, Z respectively, as indicated. Then, resolving vertically and horizontally and taking moments about E for the equilibrium of the portion $ALMNE$, we have

$$\begin{cases} W_1 + Y\cos 30^\circ = R \\ X + Y\cos 60^\circ + Z = 0 \\ R = X\sqrt{3}. \end{cases}$$

$$\therefore \begin{cases} X = \dfrac{R}{\sqrt{3}} = \dfrac{3W_1 + W_2}{4\sqrt{3}} \\ Y = \dfrac{2}{\sqrt{3}}(R - W_1) = \dfrac{W_2 - W_1}{2\sqrt{3}} \\ Z = -X - \dfrac{Y}{2} = -\dfrac{W_1 + W_2}{2\sqrt{3}} \end{cases}$$

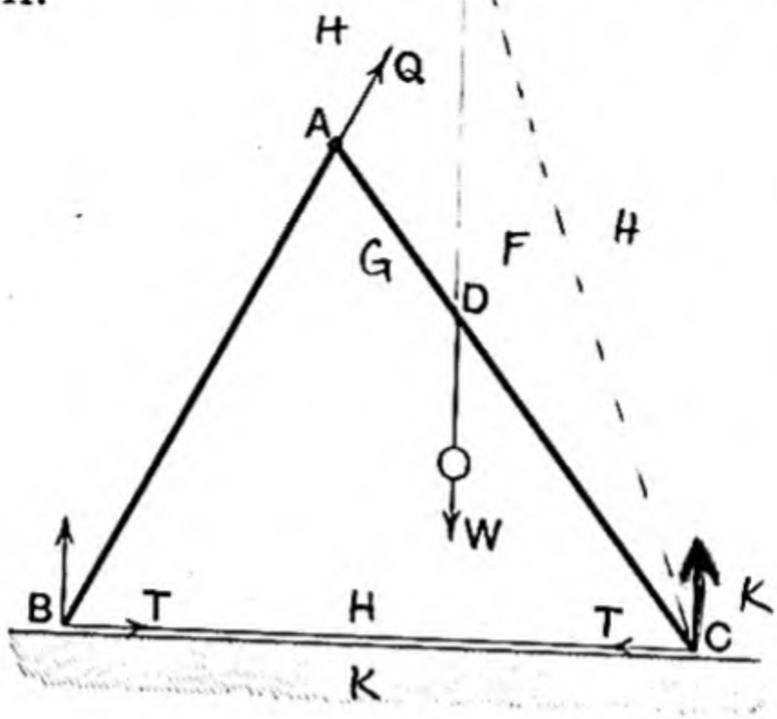
Thus

$$\begin{cases} \text{the rod } AB \text{ is a } \textit{tie}, \text{ and its tension is } \dfrac{3W_1 + W_2}{4\sqrt{3}}, \\ \text{the rod } BE \text{ is a } \textit{tie} \text{ or } \textit{strut} \text{ according as } W_2 \text{ is } > \text{ or } < W_1, \\ \text{and the stress in it is } \dfrac{W_2 \sim W_1}{2\sqrt{3}}, \\ \text{the rod } ED \text{ is a } \textit{strut}, \text{ and the thrust in it is } \dfrac{W_1 + W_2}{2\sqrt{3}}. \end{cases}$$

[For a geometrical solution of this and similar problems, see the author's *Elementary Geometrical Statics.*]

* **119.** Ex. 26.—*Two rods AB and AC, of given lengths and of no appreciable weight, are freely jointed together at A, and rest in a vertical plane upon a smooth horizontal plane at B and C, the points B and C being connected by a fine light string of given length. From a given point D of the rod AC is suspended a mass of given weight W. It is required to determine the tension of the string and the action at A.*

Let T be the tension of the string and Q the action between the rods at A.



Space diagram.

FIG. 108.

As the rod AB is in equilibrium under forces acting at A and B only, the action Q between the rods at A is in the line AB.

Thus the forces which act externally upon the rod AC are

$$\begin{cases} \text{At } A, \text{ a force } Q \text{ in direction } BA, \\ \text{At } D, \text{ a force } W \text{ vertically downwards,} \\ \text{At } C, \text{ a force } T \text{ in direction } CB, \\ \text{At } C, \text{ a pressure vertically upwards.} \end{cases}$$

A partic. case of 141[7]

As the sides of the triangle ABC are all known, we may consider the angles of this triangle as known angles. Let BC and CD be of lengths a and d respectively. Then resolving horizontally, and taking moments about C, for the equilibrium of the rod AC, we have

$$\begin{cases} T = Q \cos B \\ Q \,.\, a \sin B = W \,.\, d \cos C. \end{cases}$$

$$\therefore \begin{cases} Q = W \,.\, \dfrac{d \cos C}{a \sin B} \\ T = W \,.\, \dfrac{d \cos C}{a \tan B} \end{cases}$$

[For a solution by geometrical construction see the Author's *Elementary Geometrical Statics.*] cf. a

***120.** Ex. 27.—*Two uniform rods AB and AC, of the same material and thickness, are freely jointed together at A and to fixed*

points at B and C. If the rods are each inclined at an angle α to the vertical, while BC is inclined at an angle β to the vertical, show that the action at the hinge A is also inclined to the vertical at an angle β, and is of magnitude $\frac{1}{4}W \tan \alpha \operatorname{cosec} \beta$, *where W is the total weight of the two rods.*

Geometrical solution.—Let the vertical through A meet BC in D, and let the rods BA and AC be denoted by the figures *1* and *2* respectively.

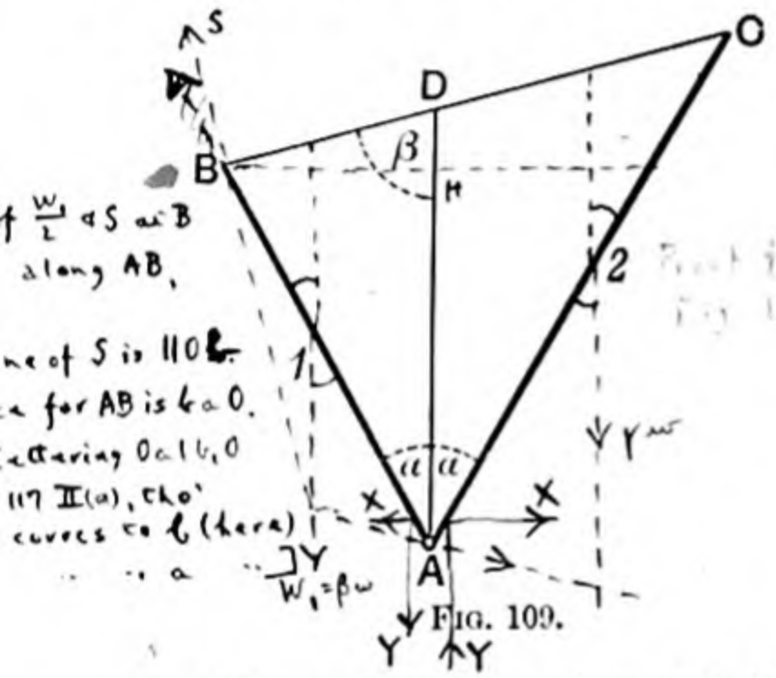


FIG. 109.

Draw *ba* and *ac* both vertically downwards, to represent the weights of the rods BA and AC respectively, and bisect *ba* and *ac* in the points *1* and *2* respectively. Draw *10* and *20* parallel to the rods *1* and *2* respectively, to meet in *0*. Then *a0* represents the action R at the hinge A.

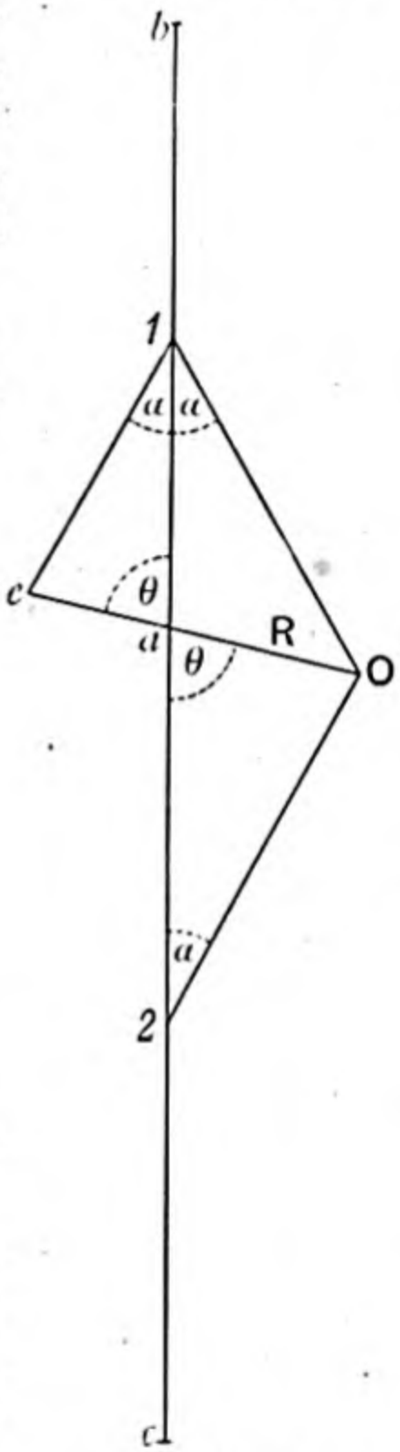


FIG. 109a.

Draw *1e* parallel to *02* or *CA*, to meet *0a* produced in *e*, and let the angle $0a2 = \theta$.

Then, as the lines *e1* and *01* are equally inclined to the vertical, we have

$$\begin{aligned} e1 : 10 &= ea : a0 \\ &= 1a : a2 \\ &= ba : ac \\ &= BA : AC. \end{aligned}$$

Also, the angle *e10* = the angle *BAC*.

Hence, the triangles $e10$ and BAC are similar, and so also are the triangles $e1a$ and BAD.

Thus $$\theta = \beta$$

Now

$$\frac{R}{W} = \frac{aO}{bc} = \frac{1}{2}\frac{aO}{12}$$

$$= \frac{1}{2}\frac{aO}{01}\cdot\frac{01}{12}$$

$$= \frac{1}{2}\frac{\sin\alpha}{\sin\theta}\cdot\frac{1}{2\cos\alpha}$$

$$= \frac{1}{4}\tan\alpha\ \mathrm{cosec}\ \beta$$

$$\therefore R = \frac{1}{4}W\tan\alpha\ \mathrm{cosec}\ \beta.$$

* **121.** Ex. 28.—*Four uniform rods, each of weight W, are freely jointed together at their extremities to form a rhombus $ABCD$. A light string connects the hinge C with a point E in AD, and the whole system is suspended from A. If each side of the rhombus is inclined to the vertical at an angle α, and if the angle DCE is of magnitude θ, show that the tension of the string is of magnitude $2W\sin\alpha\ \mathrm{cosec}\ \theta$.*

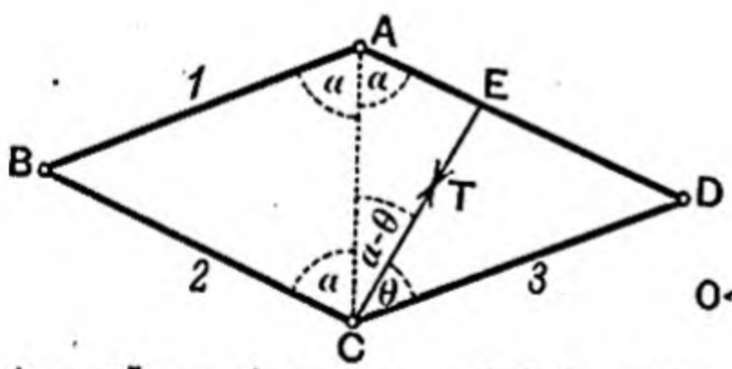


FIG. 110.

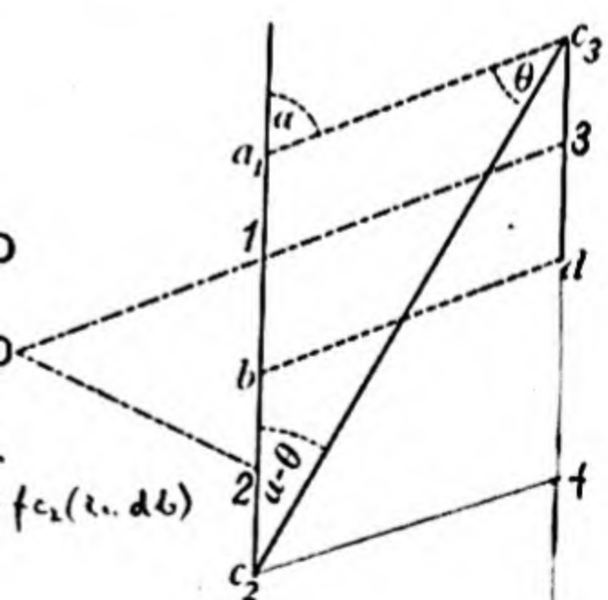


FIG. 110a.

(i) *Geometrically.*—Let the rods AB, BC, CD be denoted by the figures *1*, *2*, *3*, respectively.

Take a_1b and bc_2 vertically downwards, each of length W, and bisect a_1b and bc_2 in the points *1* and *2* respectively.

Draw *10* and *20* parallel to the rods *1* and *2* respectively, to meet in *0*.

Suppose that c_2c_3 is drawn to represent T the tension of the string, and let c_3d be drawn vertically downwards and of length W. Then, if c_3d is bisected in the point *3*, the line *03* must be parallel to the rod *3*. Thus *013* is a straight line and parallel to a_1c_3 (and bd).

Now

$$\frac{T}{2W}=\frac{c_1c_3}{a_1c_2}=\frac{\sin\alpha}{\sin\theta}$$

$$\therefore\ T=2W\sin\alpha\ \mathrm{cosec}\ \theta.$$

(ii) *Analytically.*—Let the vertical and horizontal com-

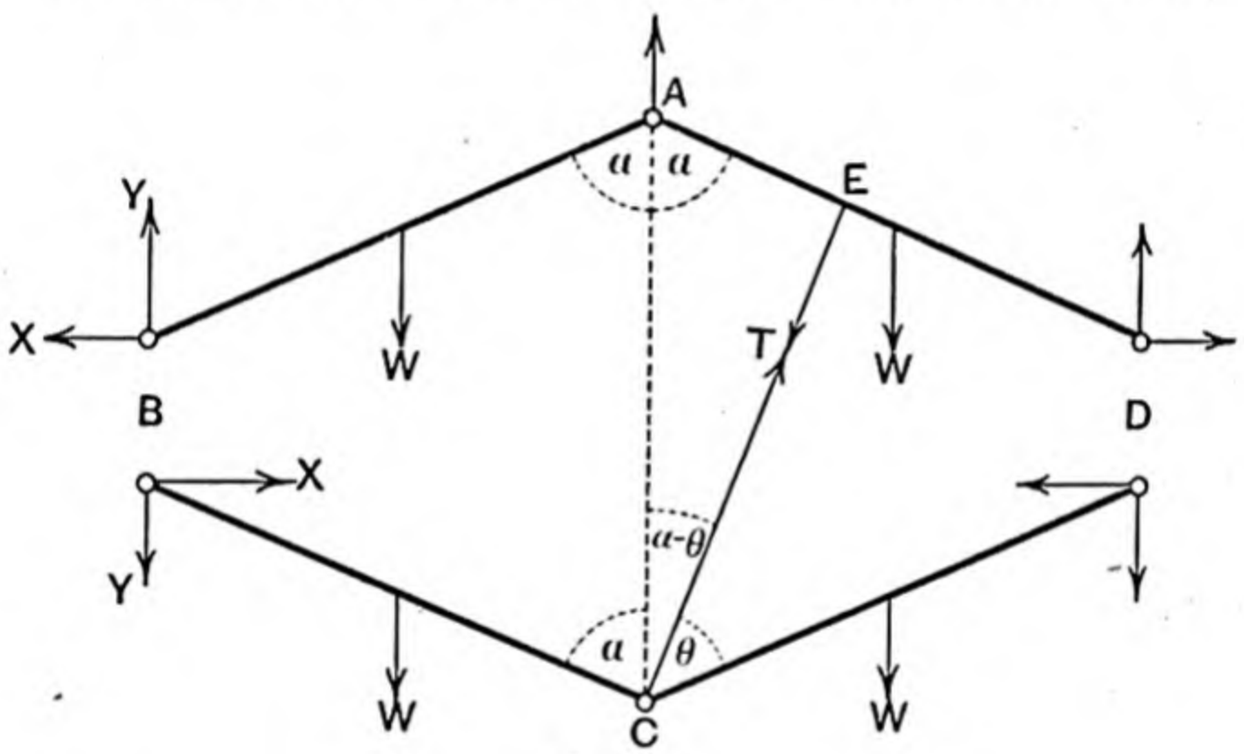


FIG. 111.

ponents of the action at B be Y and X respectively, as indicated in Fig. 111.

Taking moments about A for the equilibrium of the rod AB, we have

$$Y\,.\,2\sin\alpha+X\,.\,2\cos\alpha=W\,.\,\sin\alpha.$$

Taking moments about C for the equilibrium of the rod BC, we have

$$Y\,.\,2\sin\alpha+W\,.\,\sin\alpha=X\,.\,2\cos\alpha.$$

Thus, to find X and Y, we have the two equations

$$\begin{cases} Y\tan\alpha+X=\frac{1}{2}W\tan\alpha \\ -Y\tan\alpha+X=\frac{1}{2}W\tan\alpha. \end{cases}$$

Whence $X=\frac{1}{2}W\tan\alpha$ and $Y=0$.

Taking moments about D for the equilibrium of the system BCD, and remembering that $Y=0$, we have

$$T \cdot 2 \sin\theta = W \cdot \sin\alpha + W \cdot 3 \sin\alpha.$$
$$\therefore\ T = 2W \sin\alpha \operatorname{cosec}\theta.$$

*EXAMPLES XVI

1. Four fine light rods, each of length a, are freely jointed at their extremities to form a rhombus $ABCD$, which is placed between two parallel walls, situated at a distance $2c$ apart. The framework is in contact with the walls at A and C, and the straight line AC is perpendicular to each wall. If the joints B and D are pressed towards one another with forces each of magnitude P, find the pressure produced upon each wall.

2. Two fine light rods AB and AC, each of length a, are freely jointed together at A, and rest in a vertical plane upon a smooth horizontal plane at B and C. A mass of weight W is suspended from A, and a fine light string, of length $2l$, connects B with C. Determine the tension of the string and the thrust in each rod.

3. Three light rods are freely jointed together at their extremities to form a triangular framework ABC, the angle A being right, and the rods AB and AC of lengths b and c respectively. A mass of weight W is suspended from A, and the whole rests vertically upon a smooth horizontal plane, with which it is in contact at B and C. Determine the pressures at B and C and the stresses in the rods.

4. A light rod BC is suspended from a fixed point A by means of two light strings AB and AC. If masses of weights W_1 and W_2 are supported at B and C respectively, show that, in the position of equilibrium, the rod is inclined to the vertical at an angle

$$\cot^{-1}\left\{\frac{W_1 \cot B - W_2 \cot C}{W_1 + W_2}\right\}.$$

5. A light rod BC, of length a, is suspended from a fixed point A by means of two light strings AB and AC, of lengths c and b respectively. If masses each of weight W are supported at B and C, show that, in the position of equilibrium, the tensions of the strings are proportional to their lengths, and that the thrust in the rod is of magnitude

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}}\,W.$$

6. A light rod BC is suspended from a fixed point A by means of two equal light strings AB and AC, each inclined at an angle α to the rod. If masses of weights W_1 and W_2 are supported at B and C respectively, show that the tensions of the strings AB and AC are proportional to W_1 and W_2 respectively, and that the thrust in the rod is of magnitude

$$\frac{2W_1 W_2 \cos\alpha}{\sqrt{(W_1+W_2)^2 - 4W_1 W_2 \cos^2\alpha}}.$$

7. Three fine light rods BC, CA, AB are freely jointed together at their extremities to form a triangular framework ABC. A mass of weight W is suspended from A, and the whole is supported with BC horizontal by means of two forces applied at B and C in directions perpendicular to CA and AB respectively. Show that the tension in the rod BC is of magnitude $W \cot A$.

8. In the preceding example, if the system is supported by means of vertical forces applied at B and C, show that the thrust in the rod BC is of magnitude $W \cos B \cos C \operatorname{cosec} A$.

9. Four fine light rods are freely jointed together at their extremities to form a quadrilateral framework $ABCD$, which rests in equilibrium, with AB parallel to DC, under the influence of four forces applied at the joints. If the forces at A and D act along parallel lines, show that they are equal and in opposite directions, and that the forces at B and C are also equal and opposite.

10. A fine light rod BC, of length a, can turn freely about one extremity B. A mass of weight W is suspended from C, and a fine light string, of length $2b$, connects C with a point A situated at a distance b vertically above B. What vertical force must be applied to D, the middle point of the string, in order that the system may rest with AD parallel to BC? Determine also the thrust in the rod and the tensions of AD and DC.

11. In the preceding example, what force applied at D, in direction perpendicular to AC, will support the system in the same position of equilibrium as before?

12. A fine light string ABB_1A_1 has its extremities fixed at A and A_1 and supports masses of weight W and W_1 at B and B_1 respectively. Find the ratio of W to W_1, in order that BB_1 may be horizontal, and the angles ABB_1 and BB_1A_1 of magnitude $\pi - a$ and $\pi - a_1$ respectively.

13. A fine light string ABB_1A_1, of length $c + b + c_1$, is attached at its extremities to two points A and A_1, situated in a horizontal line at a distance a apart. Masses of weight W and W_1 are suspended from B and B_1 respectively. If BB_1 is horizontal in the position of equilibrium, and if AB and A_1B_1 are of lengths c and c_1 respectively, prove that

$$W : W_1 = \frac{(a-b)^2 + c^2 - c_1^2}{c^2} : \frac{(a-b)^2 + c_1^2 - c^2}{c_1^2}.$$

14. Four fine light rods are freely jointed together at their extremities to form a parallelogram $OBAC$, which is suspended from O. A mass of weight W is suspended from A, and horizontal forces applied at B and C keep BC horizontal and BOC a right angle. If OB and OC are of lengths b and c respectively, prove that the tensions of OB and OC are in the ratio $b^3 : c^3$, and that the action at O is of magnitude

$$W \frac{\sqrt{b^4 - b^2c^2 + c^4}}{bc}.$$

15. Four fine light rods are freely jointed together at their extremities to form a parallelogram ABA_1B_1, which is in equilibrium under the action of forces P, Q, P_1, Q_1 applied at the joints A, B, A_1, B_1 respectively. Prove that if the forces P and P_1 are equal and opposite,

then also Q and Q_1 are equal and opposite; and if the lines of action of P and Q intersect A_1B_1 in H and K respectively, then the middle point of HK coincides with the middle point of A_1B_1; also prove that $P : Q = AH : KB$.

16. A light rod BC, of length a, is capable of turning freely about one extremity B, which is fixed. A fine light string, of length $a+c$, connects C with a point A, situated at a distance c vertically above A. The whole is supported, with BC horizontal and a mass of weight W suspended from C, by means of a vertical force applied at D, a point of the string situated at a distance c from A. Show that the force applied at D is of magnitude

$$W\left(\frac{a^2+c^2}{2ac}\right)^2.$$

17. In the preceding example, if the force applied at D is in direction BD, show that it must be of magnitude

$$W\frac{(a^2+c^2)^{\frac{3}{2}}}{2ac^2},$$

and that the tension of AD is increased in the ratio $a^2-c^2 : 2a^2$.

18. Four light rods are freely jointed together at their extremities to form a parallelogram $OBAC$, the hinges O and A being connected by a light string. If masses, each of weight W, are suspended from B and C, and the whole system is supported at O, determine the tension of the string and the stresses in the rods, supposing that OA, OB, OC are of lengths a, b, c respectively.

19. Four light rods are freely jointed together at their extremities to form a quadrilateral $OBAC$, the rods OB and OC being each of length b, and AB and AC each of length c, and the hinges A and O being connected by means of a fine light string of length a. If masses each of weight W are suspended from B and C, the whole system being supported at O, determine the tension of the string.

20. Four light rods are freely jointed together at their extremities to form a rhombus $OBAC$, the hinges O and A being connected by means of a fine light string. Masses of weight W_1 and W_2 are suspended from B and C respectively, and the whole system is supported at O. If the string is inclined to each rod at an angle α, show that the system rests with OA inclined to the vertical at an angle

$$\tan^{-1}\left\{\frac{W_1 \sim W_2}{W_1+W_2}\cdot\tan\alpha\right\},$$

and that the tension of the string is of magnitude

$$\frac{2W_1W_2\cos\alpha}{\sqrt{(W_1\sim W_2)^2+4W_1W_2\cos^2\alpha}}.$$

21. Three light rods AB, BC, CD are freely jointed together at B and C and to fixed points at A and D, and another light rod connects the hinges A and C. If AD and BC are both horizontal, and BC below AD, and if ACD is a right angle, while the angles DAB and ADC are each of magnitude α, determine the stresses in AC and CD when masses of weight W_1 and W_2 are suspended from B and C respectively.

22. Four light rods are freely jointed together at their extremities to form a quadrilateral framework $ABCD$, the hinges A and C being connected by another light rod. The rods AD and BC are parallel and ACD is a right angle, and the angles DAB and ADC are each of magnitude α. If the whole rests vertically upon a smooth horizontal plane at A and D, determine the pressures at A and D when masses of weight W_1 and W_2 are suspended from B and C respectively.

23. Four light rods are freely jointed together at their extremities to form a parallelogram $OBAC$, the hinges O and A being connected by another light rod. A body of weight W is suspended from O, and the whole system is supported with AO vertically downwards by means of vertical forces applied at B and C. Determine the values of these forces and the stresses in the rods, supposing that OA, OB, OC are of lengths a, b, c respectively.

24. In the preceding example, the whole system is supported, with O vertically below the middle point of AC, by means of vertical forces applied at B and C. Determine the values of these forces and the stresses in the rods.

25. $OBAC$ is a quadrilateral framework of four light rods freely jointed together at their extremities, the hinges A and O being connected by another light rod. A mass of weight W is suspended from O, and the whole is supported with AO vertically downwards by means of vertical forces applied at B and C. Prove that if the tension of AO is of magnitude T, and if BC intersects AO in the point D, then

$$T : W = AD : AO.$$

26. $OBAC$ is a quadrilateral framework of four light rods freely jointed together at their extremities, the hinges A and O being connected by another light rod, which passes through D the middle point of BC. A mass of weight W is suspended from O, and the whole is supported with AO vertically downwards by means of forces applied at B and C in directions parallel to CA and BA respectively. Prove that if the tension of AO is of magnitude T, then

$$W - T : 2W = AD : AO.$$

27. $OABB_1A_1$ is a framework of seven light rods freely jointed together at their extremities. The rods OB and OB_1 are equal; the rods AB and OB_1 are parallel, and so are A_1B_1 and OB; the angles A and A_1 are right, and the angle BOB_1 is of magnitude α. If masses each of weight W are suspended from A and A_1, and if the whole system is supported at O, determine the thrust in the rod BB_1.

28. In the preceding example, if masses of weight W and W_1 are suspended from A and A_1 respectively, show that the thrust in BB_1 is of magnitude

$$\frac{2WW_1 \sin\alpha \sin\frac{\alpha}{2}}{\sqrt{W^2 + W_1^2 - 2WW_1 \cos\alpha}}.$$

29. Two light rods AB and AC, each of length a, are freely jointed together at A, and rest in a vertical plane upon a smooth horizontal

plane at B and C, the points B and C being connected by a fine light string of length $2l$. From a point D of the rod AC is suspended a mass of weight W. If CD is of length d, prove that the tension of the string is of magnitude

$$\tfrac{1}{2}W\frac{ld}{a\sqrt{a^2-l^2}}.$$

30. Two light rods AB and CAD are freely jointed together at A. A fine light string connects B and C, and the whole rests in a vertical plane upon a smooth horizontal plane at B and C, a mass of weight W being suspended from D. If BC and CD are of lengths a and d respectively, prove that d must be less than a sec C, and that the tension of BC and the thrust in AB are respectively of magnitude

$$W\frac{d}{a}\cos C\cot B \quad \text{and} \quad W\frac{d}{a}\cos C\operatorname{cosec} B.$$

31. Two equal rods AB and AC, of no appreciable weight, are freely jointed together at A, and rest in a vertical plane upon a smooth horizontal plane at B and C, a light string connecting a point E in the rod AB with a point F in the rod AC. The rod AC is loaded at a point X with a mass of weight W. Prove that, whatever be the position of X in AC, the action between the rods at A is in a fixed direction, and that the pressure at B, the tension of the string, and the action at A, all vary as CX. Show also that if AM and XY are drawn perpendicular upon EF and BC respectively, and if the tension of the string is of magnitude T, then

$$T:\tfrac{1}{2}W=CY:AM.$$

32. In the preceding example, if the rods are not of equal length, and if AN is drawn perpendicular upon BC, then

$$T:W=BN.CY:AM.BC.$$

33. In Example 32, if F is the middle point of AC and EF perpendicular to AB, and if the angle A is 60°, prove that the action at A is horizontal, and that, for different positions of X in AC, the tension of the string varies between 0 and W.

34. Two uniform rods AB and AC, of weight W_1 and W_2 respectively, are freely jointed together at A, and rest in a vertical plane upon a smooth horizontal plane at B and C, a light string connecting the points E and F which divide AB and AC respectively, each in the ratio 2 : 1. If AB and AC are inclined at angles B and C respectively to the horizontal, prove that the tension of the string is of magnitude

$$\frac{3}{4}\left(\frac{W_1+W_2}{\tan B+\tan C}\right).$$

Show also that the action at A is inclined to the horizontal at an angle

$$\tan^{-1}\left\{\frac{2}{3}\left(\frac{W_1\tan C-W_2\tan B}{W_1+W_2}\right)\right\}.$$

35. Four light rods are freely jointed together at their extremities to form a rhombus $OBAC$, which is stiffened by another light rod freely jointed to the middle points of OB and OC. A mass of weight

W is suspended from A, and the whole framework is supported at O. Prove that if each side of the rhombus is inclined to the vertical at an angle α, then the thrust in the cross rod is of magnitude

$$2W \tan \alpha.$$

36. Four light rods are freely jointed together at their extremities to form a rhombus $OBAC$, which is stiffened by another light rod connecting B with the middle point of AC. A mass of weight W is suspended from A, and the whole framework is supported at O. Prove that if each side of the rhombus is inclined to the vertical at an angle α, then the thrust in the cross rod is of magnitude

$$\tfrac{1}{2}W\sqrt{1+9\tan^2\alpha}.$$

37. Two uniform rods AB and BC, of weight W_1 and W_2 respectively, are freely jointed together at B, and the former rod can turn freely about its extremity A, which is fixed. The whole is supported, with ABC horizontal, by means of a vertical force applied to the rod BC. Show that this force is of magnitude $W_2+\frac{1}{2}W_1$, and that its point of application divides BC in the ratio $W_2 : W_1+W_2$.

38. Four uniform rods, each of weight W, are freely jointed together at their extremities to form a rhombus $ABCB'$. A light string connects the hinges A and C, and the whole system is suspended from A, the rods being inclined to the vertical each at an angle α. Show that the tension of the string is $2W$, and that the actions at B and B' are each $\frac{1}{2}W \tan \alpha$ in a horizontal direction.

39. Four uniform beams, each of weight W, are freely jointed together at their extremities to form a rhombus $ABCB'$. A light rod connects the hinges B and B', and the whole system is suspended from A, the beams being inclined to the horizontal each at an angle α. Show that the thrust in BB' is of magnitude $2W \cot \alpha$, and that the action at C is $\frac{1}{2}W \cot \alpha$ in a horizontal direction.

40. Three uniform rods AB, BB', $B'A'$, of weight W, w, W respectively, the two AB and $A'B'$ being equal in length, are freely jointed together at B and B' and laid in a vertical plane upon a smooth horizontal table, a light string connecting a point C in the rod AB with a point C' in the rod $A'B'$. Prove that, if BC and $B'C'$ are each $\frac{1}{n}$th of AB, and if AB and $A'B'$ are each inclined to the horizontal at an angle α, then the tension of the string is of magnitude

$$\tfrac{1}{2}n(W+w)\cot\alpha.$$

41. Three uniform rods AB, BB', $B'A'$, of weight W, w, W respectively, are freely jointed together at B and B', and suspended from a fixed point O by means of four light strings OA, OB, OB', OA' respectively. The system rests with OB and OB' each inclined at an angle α to the vertical, OA and OA' each inclined at an angle α to the horizontal, BB' horizontal, and the angles A and A' right angles, the whole being symmetrical about the vertical through O. Show that

(i) The tensions of OA and OA' are each $\frac{1}{2}W \sin \alpha$;

(ii) The tensions of OB and OB' are each $\frac{1}{2}(W+w)\sec\alpha+\frac{1}{2}W\cos\alpha$.

42. Three uniform rods AB, BB', $B'A'$, of weight W, w, W respectively, are freely jointed together at B and B', and suspended from a

fixed point O by means of three light strings OA, OC, OA', the point C being the middle point of BB'. The system rests with OC vertical, BB' horizontal, the angles A and A' right angles, and the strings OA and OA' each inclined at an angle α to OC, the whole being symmetrical about OC. Show that

(i) The tensions of OA and OA' are each $\frac{1}{2}W\cos\alpha$;

(ii) The tension of OC is $W(1+\sin^2\alpha)+w$;

(iii) The actions at B and B' are each $\frac{1}{2}W\sqrt{1+3\sin^2\alpha}$ at an angle $\tan^{-1}(2\tan\alpha+\cot\alpha)$ with the horizontal.

43. Two uniform rods AB and AC, of the same material and thickness, are freely jointed together at A, and to fixed points at B and C. The bisector of the angle BAC meets BC in D, and through D two straight lines are drawn, one parallel to CA to meet the vertical through B in B', the other parallel to BA to meet the vertical through C in C'. Prove that $B'C'$ passes through A, and is the line of action of the strain upon the hinge A; also, the lines of action of the strains upon the hinges B and C bisect AB' and AC' respectively.

44. In the preceding question, if BC is vertical, prove that the action at A is along the bisector of the angle BAC.

45. Two uniform rods AB and AC are freely jointed together at A, and to fixed points B and C situated in a vertical line. Prove that, if D divides BC in the ratio of the weights of BA and AC, and if E is the middle point of AD, then the actions at A, B, C are in the lines EA, EB, EC respectively. Also these actions and the total weight of the two rods are respectively proportional to EA, EB, EC, BC.

46. Two equal uniform rods AB and AC, each of weight W, are freely jointed together at A, and suspended from a fixed point O by means of three light strings OA, OB, OC. When the strings are tight, the rods BA and AC are in a straight line and BOC is a right angle, the angle OBA being of magnitude α. If BAC is horizontal when a mass of weight X is suspended from C, show that

$$X = W(\cot^2\alpha - 1),$$

and that the tensions of the strings OA, OB, OC are respectively

$$W\operatorname{cosec} 2\alpha,\ \tfrac{1}{2}W\operatorname{cosec}\alpha,\ \tfrac{1}{2}W\sec\alpha\,(2\cot^2\alpha - 1).$$

47. Four uniform rods, each of weight W, are freely jointed together at their extremities to form a rhombus $ABCD$. The rod AD is held in a horizontal position and uppermost, and a light string connects the hinges A and C, so that the angle ABC is of magnitude α. Show that the tension of the string is of magnitude

$$2W\cos\alpha\sec\tfrac{1}{2}\alpha.$$

48. In the preceding example, if the string is equal in length to one of the rods, show that the action at D is $\frac{1}{2}W$ in a vertical direction.

49. In Ex. 28, Art. 121, if E is the middle point of AD, show that the tension of the string is of magnitude $W\sqrt{9+\tan^2\alpha}$, and that the action at D is of magnitude $W\sqrt{1+\frac{9}{4}\tan^2\alpha}$, and is inclined to the vertical at an angle $\tan^{-1}(\frac{3}{2}\tan\alpha)$.

50. Two equal uniform beams AB and AB', each of weight W, are freely jointed together at A and suspended from a fixed point O by means of two light strings OB and OB' of equal length. A light rod is freely jointed to the beams at C and C', which divide AB and AB' respectively each in the ratio $2:1$. Show that, if the strings are each inclined at an angle α to the vertical, and the beams at an angle β, then the thrust in the rod is of magnitude $\frac{3}{4}W(2\tan\alpha+\tan\beta)$, and the action at A is horizontal and equal to $\frac{1}{4}W(2\tan\alpha+3\tan\beta)$.

51. Six uniform rods, each of weight W, are freely jointed together at their extremities to form a regular hexagon $ABCC'B'A'$. The whole rests in a vertical position with AA' on a smooth horizontal plane, the shape being preserved by means of a light string connecting the middle points of BC and $B'C'$. Show that (i) the tension of the string is $2W\sqrt{3}$, (ii) the actions at B and B' are each $\frac{1}{6}W\sqrt{129}$, (iii) the actions at C and C' are each $\frac{1}{6}W\sqrt{201}$.

52. Two uniform rods AB and AC, each of weight W, are freely jointed together at A, and suspended from a fixed point O by means of three light strings OA, OB, OC, so that the figure $OBAC$ is a rhombus with the angles B and C each of magnitude α. If AC is horizontal when a mass of weight X is suspended from C, show that $X=\frac{3}{2}W(\sec\alpha-1)$, and that the tensions of OA, OB, OC, are respectively $\frac{3}{2}W\sec\frac{1}{2}\alpha$, $\frac{1}{2}W\cot\alpha$, $W(3\operatorname{cosec}2\alpha-\operatorname{cosec}\alpha)$.

53. Four equal uniform rods, each of weight W, are freely jointed together at their extremities to form a rhombus $ABCD$. A fine light string is attached to the rods AD and CD at points which divide AD and CD each in the ratio $m:n$. The whole system is supported at A, and each rod is inclined to the vertical at an angle α. Show that the tension of the string is of magnitude $\frac{m+n}{n}.2W$, and that the actions at B, C, D are respectively of magnitude $\frac{1}{2}W\tan\alpha$, $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$, $\frac{1}{2}W\sqrt{16\frac{m^2}{n^2}+\tan^2\alpha}$.

54. Three equal uniform rods AB, BC, CD, each of weight w, are freely jointed together at B and C, and rest in a vertical plane upon a smooth horizontal plane at A and D. Two equal light strings AC and BD help to support the framework, so that AB and CD are each inclined at an angle α to the horizontal. Show that if a mass of weight W is placed on BC at its middle point, then the tension of each string is of magnitude

$$(w+\tfrac{1}{2}W)\cos\alpha\operatorname{cosec}\tfrac{1}{2}\alpha.$$

55. In the preceding example, if the mass is moved along the rod BC through a distance equal to $\frac{1}{n}$th of BC, show that the tension of one string is increased by an amount

$$\frac{W}{n\sin\frac{\alpha}{2}(2+\sec\alpha)}$$

and that the tension of the other is diminished by the same amount.

56. Two equal uniform rods CB and CB', each of length a and weight W, are freely jointed together at C, and connected with a fixed point A by means of three light strings AB, AC, AB', of lengths c, b, c respectively. Prove that, in the position of equilibrium, the tensions of AB and AB' are each of magnitude $\frac{1}{2}W\frac{c}{b}$, and that the tension of AC is of magnitude

$$\tfrac{3}{2}W+\frac{a^2-c^2}{2b^2}W.$$

57. Two equal uniform rods AB and AC, each of weight w, are freely jointed together at A, and suspended from a fixed point O by means of three light strings OA, OB, OC. Masses each of weight W are suspended from B and C, and the strings are of such length that BAC is a straight line and BOC a right angle, the angle OBC being of magnitude α. Prove that, in the position of equilibrium, the tensions of the strings OA, OB, OC are respectively w, $(w+2W)\cos\alpha$, $(w+2W)\sin\alpha$.

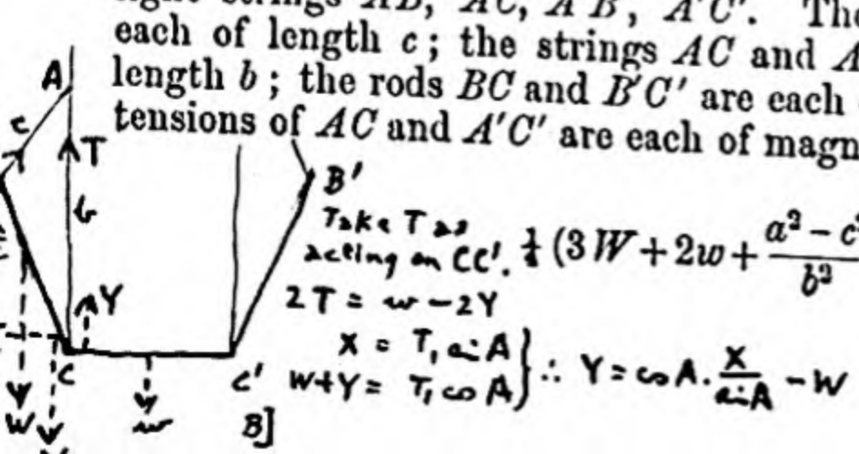
58. Four uniform beams are freely jointed together at their extremities to form a quadrilateral framework $ABCB'$, the beams AB and AB' being of equal length and each of weight w, and the beams CB and CB' of equal length and each of weight W. A light rod connects the hinges B and B', and the whole system is suspended from A. If the upper rods are each inclined to the horizontal at an angle α, and the lower rods each inclined to the horizontal at an angle β, show that the thrust in BB' is of magnitude

$$W\cot\alpha+\tfrac{1}{2}(w\cot\alpha+W\cot\beta).$$

59. Four uniform rods are freely jointed together at their extremities to form a quadrilateral framework $ABCB'$, the rods AB and AB' being each of length c and weight w, and the rods CB and CB' each of length a and weight W. A light string, of length b, connects the hinges A and C, and the whole system is suspended from A. Prove that the tension of the string is of magnitude

$$\tfrac{1}{2}(3W+w)+\frac{a^2-c^2}{2b^2}(W+w).$$

60. Three uniform rods BC, CC', $C'B'$, of weights W, w, W respectively, are freely jointed together at C and C' and suspended from two fixed points A and A', situated in a horizontal line, by means of four light strings AB, AC, $A'B'$, $A'C'$. The strings AB and $A'B'$ are each of length c; the strings AC and $A'C'$ are vertical and each of length b; the rods BC and $B'C'$ are each of length a. Show that the tensions of AC and $A'C'$ are each of magnitude



61. Four uniform rods, of the same material and thickness, are freely jointed together at their extremities to form a parallelogram $ABCB'$, the rods AB and BC being of lengths c and a respectively. A light string of length b connects the hinges A and C, and the whole system, which is of weight W, is suspended from A. Show that, in the position of equilibrium, the tension of the string is $\frac{1}{2}W$, and that the action at B bisects the angle ABC and is of magnitude

$$\frac{1}{4}W\sqrt{\frac{ca}{b^2}-\frac{ca}{(c+a)^2}}.$$

CHAPTER VIII

PARALLEL FORCES—CENTRE OF GRAVITY

122. Resultant of Parallel Forces.—*To find the resultant of a given system of parallel forces acting in one plane upon a rigid body.*

Let O be a fixed point in the plane of the forces, and OX a straight line drawn from O perpendicular to the lines of action of the forces. Let P be the measure of one of the forces of the system, and p the distance from O of the point where its line of action meets OX, so that p is negative if the line of action of P meets XO produced.

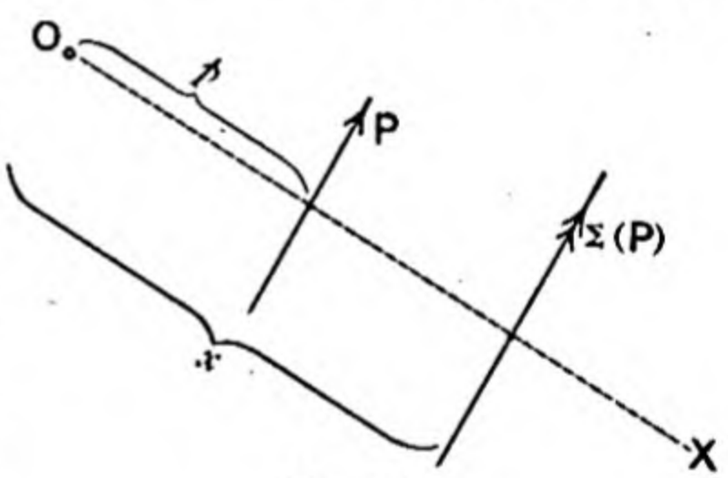


FIG. 112.

The force polygon evidently reduces to a straight line. Hence, regarding forces which act in one direction as positive and those which act in the other direction as negative, we see that the resultant is equal to the algebraic sum of the forces, and is parallel to each of the component forces. Thus, if R is the measure of the resultant, we have

$$R = \Sigma(P).$$

Let x be the distance from O of the point where the line of action of the resultant meets OX. Then, since the moment of the resultant about O is equal to the sum of the moments of its components about O, we have

$$Rx = \Sigma(Pp),$$

$$\therefore x = \frac{\Sigma(Pp)}{\Sigma(P)}.$$

If the force polygon closes, we have $\Sigma(P)=0$. In this case the forces are equivalent to a couple of moment $\Sigma(Pp)$, unless $\Sigma(Pp)$ also vanishes, in which case, as we shall see in the next article, the forces are in equilibrium.

Example.—If R is the resultant of the four parallel forces 2, 3, 4, 5 indicated in Fig. 113, we have

$$R=3+5-4-2=2$$

and
$$Rx\sin\alpha=5\times4\sin\alpha-4\times7\sin\alpha+2\times3\sin\alpha,$$
$$\therefore 2x=-2,$$
$$\therefore x=-1.$$

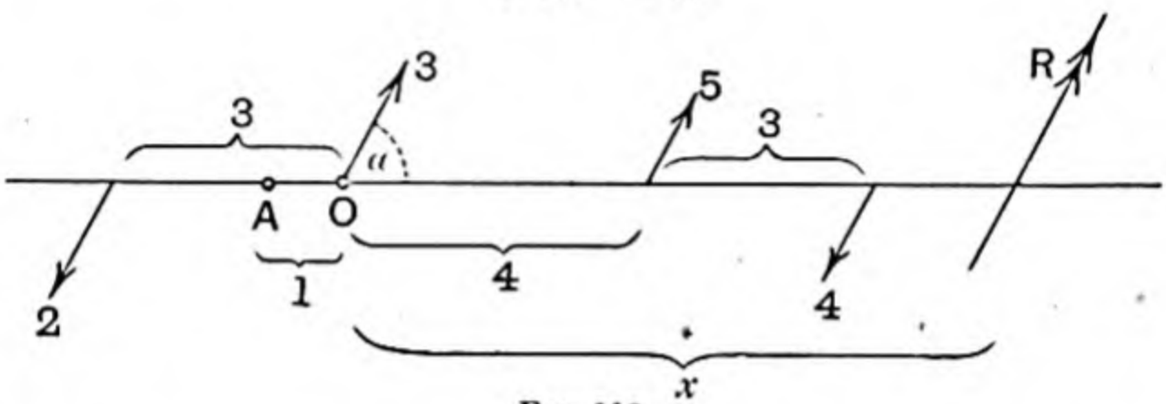


FIG. 113.

Thus the resultant is of magnitude 2 and acts through the point A in the direction of the forces 3 and 5.

* **123. Conditions of Equilibrium.**—In considering the general case of a system of parallel forces as in the preceding article, we notice that one of the conditions of equilibrium of Art. 106 is identically satisfied; namely, the sum of the resolved parts of the forces in the direction OX is identically zero. We are thus left with only two conditions of equilibrium; namely, the sum of the resolved parts of the forces in one direction (not perpendicular to their lines of action) must be zero, and the sum of the moments of the forces about one point must be zero.

Thus, for equilibrium, it is necessary and sufficient that

$$\begin{cases}\Sigma(P)=0\\ \text{and } \Sigma(Pp)=0.\end{cases}$$

* **124.** Again, let O' be another fixed point in the plane of the forces, *such that OO' is not parallel to the lines of action of the forces*, and let $O'X'$ be drawn perpendicular to their lines of action. Let p' be the distance from O' of the point where the line of action of P meets $O'X'$. Then, as the sum of the resolved parts of the forces in the direction

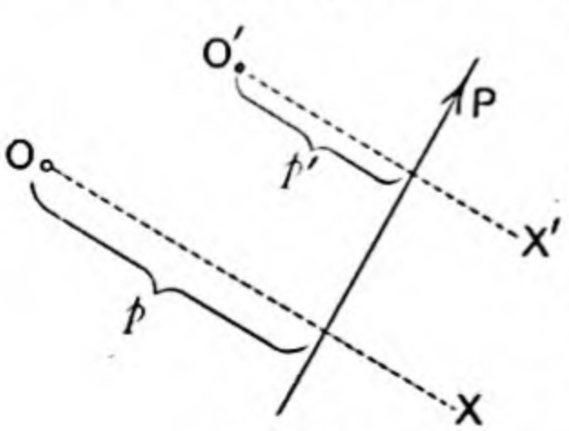


FIG. 114.

OX, which is not perpendicular to OO', is identically zero, it follows by Art. 108 that it is necessary and sufficient for equilibrium that the sums of the moments of the forces about *O* and *O'* should be separately zero.

Thus, for equilibrium, it is necessary and sufficient that

$$\begin{cases} \Sigma(Pp) = 0 \\ \text{and } \Sigma(Pp') = 0. \end{cases}$$

**125. We see then that two conditions are necessary and sufficient for the equilibrium of a system of parallel forces in one plane. In dealing with the equilibrium of any such system, we can write down two independent equations. We either resolve in the direction of the forces and take moments once, or we take moments twice.

*126. Ex. 29.—*A straight lever AB, of weight w, balances about a fulcrum F when masses of weight W and W' are suspended from A and B respectively. If W and W' are interchanged, show that the fulcrum must be moved through a distance equal to* $\frac{W - W'}{W + W' + w}$ *of the length of the lever.*

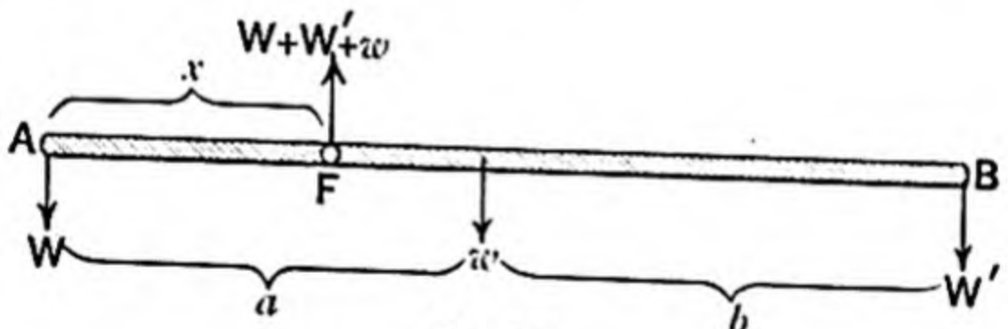


FIG. 115.

Resolving vertically upwards, we see that the pressure of the fulcrum upon the lever is $W + W' + w$.

Let the fulcrum be at a distance x from A, and let the centre of gravity of the beam be at distances a and b from A and B respectively.

Then, taking moments about A, we have

$$(W + W' + w)x = wa + W'(a + b),$$

$$\therefore x = \frac{wa + W'(a + b)}{W + W' + w}.$$

Now let x become y when W and W' are interchanged.

Then

$$y = \frac{wa + W(a + b)}{W' + W + w}.$$

The distance through which the fulcrum must be moved is $y - x$

$$= \frac{W - W'}{W + W' + w}(a + b).$$

*** 127.** Ex. 30.—*A mass of weight W is placed on a heavy beam of weight w, which is supported in a horizontal position by two vertical strings attached to its extremities. If each string can withstand a tension not greater than a given value P, show that the mass must be placed on a certain part of the beam of length equal to* $\frac{2P - W - w}{W}$ *of the length of the beam.*

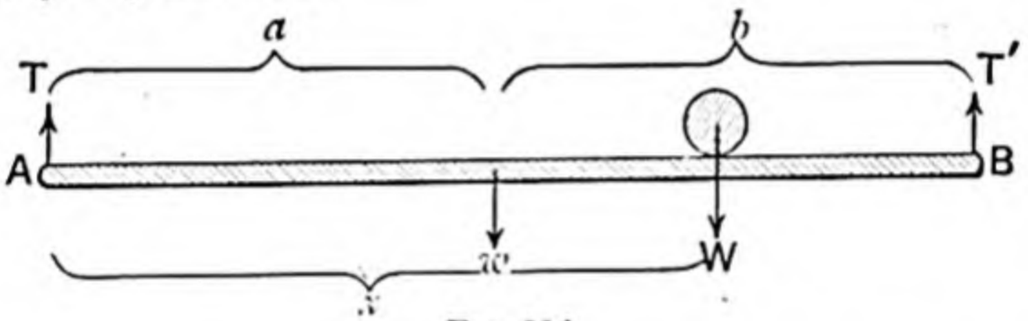


FIG. 116.

Let AB be the beam, and let its centre of gravity be at distances a and b from A and B respectively.

Let T and T' be the tensions of the strings at A and B respectively when the mass of weight W is placed at a distance x from A.

Then, taking moments about A, we have $T'(a + b) = Wx + wa$.

Now T' must not be greater than P.

$\therefore$ $Wx + wa$ must not be greater than $P(a + b)$.

$\therefore$ x " " $\frac{P}{W}(a + b) - \frac{w}{W}a$.

Thus the mass must not be placed at a distance from A greater than $\frac{P}{W}(a + b) - \frac{w}{W}a$.

Similarly it must not be placed at a distance from B greater than $\frac{P}{W}(a + b) - \frac{w}{W}b$.

The part of the beam on which the mass may be placed is of length

$$2\left[\frac{P}{W}(a + b) - \frac{w}{W}a\right] + \left[\frac{P}{W}(a + b) - \frac{w}{W}b\right] - (a + b)$$

$$= \frac{2P - W - w}{W}(a + b).$$

EXAMPLES XVII

1. Two unlike parallel forces P and $4P$ act on a rigid body; and, midway between them, and in the same direction as the greater force, a third force $7P$ acts. Find the magnitude and line of action of the resultant of the three forces.

2. ABC is a straight line, AB and BC being of lengths 2 and 4 inches respectively. Three parallel forces of 4, 3, 5 pounds' weight act at the points A, B, C respectively, the first force being in the opposite direction to each of the other two. Determine the resultant.

3. A uniform rod ACB, of weight w, is supported, in a horizontal position, by a vertical string fastened to its middle point C. From the end B a mass of weight w is suspended; and from a point D in CA a mass of weight Q is suspended. Find Q, and the position of the point D, when the tension of the string at the middle point is $4w$.

4. The extremities A and B of a uniform rod rest on two supports, while a mass of 30 pounds is attached at a point C, such that AC is twice CB. Find the pressure at A, (i) when the rod is light, (ii) when it weighs 10 pounds.

5. A uniform beam AB, of length 16 feet and mass 25 pounds, rests in a horizontal position upon two smooth supports at its extremities. It is loaded at points distant 5, 9, 12 feet from A with 100, 40, 61 pounds respectively. Find the pressures on the supports.

6. A bookshelf, supported at its extremities, is just filled by two sets of books, the books of each set being placed together. One set consists of 14 volumes, each $1\frac{1}{2}$ inches thick, and each weighing $2\frac{1}{2}$ pounds; the other consists of 12 volumes, each $1\frac{1}{4}$ inches thick, and each weighing 2 pounds. Find the pressures on the supports, the mass of the shelf being 8 pounds.

7. A uniform rod AB, of length 1 foot and mass 42 pounds, rests upon two pegs whose distance apart is equal to half the length of the rod. How must it be placed so that the pressure on the pegs may be equal when a mass of 14 pounds is suspended from B.

8. A uniform lever AB balances about a fulcrum F, when masses of 15 and 14 pounds are suspended from A and B respectively. The mass suspended from A is replaced by 45 pounds, and it is then found that the fulcrum has to be moved from F to the middle point of AF. Find the mass of the lever.

9. A bent lever, of weight w, consists of two uniform straight rods, of the same material and thickness, and whose lengths are as 3 to 4, the fulcrum being at the junction of the rods. Find the weight of the mass which must be attached to the end of the shorter rod, in order that the rods may make equal angles with the horizontal.

10. A uniform straight rod, of length 18 inches, and weighing 9 pounds, rests in a horizontal position upon two smooth supports at its extremities. If neither support can sustain a pressure greater than 50 pounds' weight, find the greatest load which may be applied to the rod at a point 5 inches from one end.

11. A heavy uniform beam, of length 20 feet and weighing 50 pounds, rests in a horizontal position upon two smooth supports at its

extremities. If neither support can sustain a pressure greater than 40 pounds' weight, how far from the centre of the beam must a mass of 20 pounds be placed, so that one support may just give way?

12. A uniform rod, of length 1 foot and mass 10 pounds, rests in a horizontal position upon two smooth supports at its extremities. If neither support can sustain a pressure greater than 26 pounds' weight, how far from the centre of the rod must a mass of 28 pounds be placed, so that one support may just give way?

13. A beam AB, of mass 45 pounds, rests in a horizontal position upon two smooth supports at its extremities. If the centre of gravity of the beam is 5 feet from A and 4 feet from B, find the pressures on the supports when a mass of 33 pounds is placed on the beam at a point 3 feet from A.

14. In the preceding question, where must the mass be placed in order that the tensions of the strings may be equal to one another?

15. If, in question 13, it is desirable that neither support should have to sustain a pressure greater than 50 pounds' weight, show that there is a portion of the beam of length 6 feet on which the mass may be placed.

16. A straight lever AB, of weight w, balances about a fulcrum F when masses of weight W and W' are suspended from A and B respectively. If W is increased by P and W' diminished by the same amount, show that the fulcrum must be moved through a distance equal to $\dfrac{P}{W+W'+w}$ of the length of the lever.

17. Masses of weight W and W' are suspended from the extremities A and B of a uniform rod AB. If the rod balances about a point distant a and b from A and B respectively, show that its weight is

$$\frac{2(Wa-W'b)}{b-a}$$

18. A straight rod AB, of length 1 foot, balances about a point 5 inches from A when masses of 9 and 6 pounds are suspended from A and B respectively; it balances about a point $3\frac{1}{2}$ inches from B when the mass of 6 pounds is replaced by one of 23 pounds. Find the mass of the rod and the position of its centre of gravity.

19. A bent lever, of weight w, consists of two uniform straight rods, of lengths a and b and of the same material and thickness, the fulcrum being at the junction of the rods. Find the weight of the mass which must be attached to the end of the shorter rod b, in order that the rods may make equal angles with the horizontal.

20. A bent lever, of weight w, consists of two uniform straight rods, of lengths a and b and of the same material and thickness, inclined to one another at an angle 2α, the fulcrum being at the junction of the rods. If masses of weight W and W' are suspended from the ends of the arms a and b respectively, show that the bisector of the angle between the arms is inclined to the vertical at an angle

$$\tan^{-1}\left\{\left(\frac{2(Wa-W'b)+(a-b)w}{2(Wa+W'b)+\dfrac{a^2+b^2}{a+b}w}\right)\tan\alpha\right\}.$$

128. Centre of Parallel Forces.—Suppose that a number of parallel forces $P_1, P_2, P_3, \ldots$ act *in the same direction* at points $A_1, A_2, A_3, \ldots$ of a rigid body.

The two forces P_1 and P_2 have a resultant $P_1 + P_2$, acting in the same direction as each of its components along a line which intersects A_1A_2 at the point B, where $A_1B : BA_2 = P_2 : P_1$.

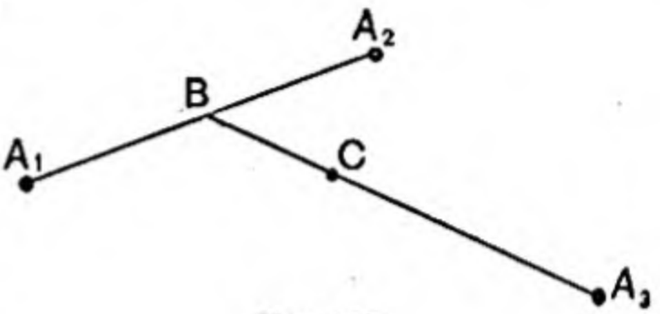


FIG. 117.

Replacing the forces P_1 and P_2 by their resultant, and combining this with the next force P_3, we see that the forces P_1, P_2, P_3 can be replaced by a single force $P_1 + P_2 + P_3$, acting in the same direction as before along a line which intersects BA_3 at the point C, where $BC : CA_3 = P_3 : P_1 + P_2$.

Proceeding in this way, we see that the whole system is equivalent to a single force $\Sigma(P)$, acting in the same direction as each of its components through a point G, whose position depends only upon the positions of the points $A_1, A_2, A_3, \ldots$ and the relative magnitudes of the forces.[1] The directions of the forces may be altered, provided that they remain parallel and in the same direction,[2] and the magnitudes of the forces may be altered, provided they are all increased or diminished in the same ratio, and the resultant still passes through the same point G.

[3] The order in which the forces are taken is of no importance, for we have seen in Art. 98 that in no instance can two different resultants be obtained to the same system of forces.

The point G is called the *centre of the system of parallel forces.*

129. Centre of Mass.—In considering any assemblage of particles, there exists a point called the *Centre of Mass* of the system, which possesses the property that if each particle were acted upon by a force proportional to its mass, all the forces being parallel to one another and in the same direction, then the resultant of the system of forces would always pass through that point, whatever be their common direction.

130. Centre of Gravity.— We have referred in the first chapter to the nature of the forces constituting the weight of a body. In the case of a rigid body of small size in comparison with the distances of its different parts from the centre of the

earth, these forces may be regarded as a system of parallel forces acting on the different particles of which the body is composed, and respectively proportional to the masses of those particles. If the body is turned about in any way, the directions of the forces remain the same as before relatively to the earth, but assume different directions relatively to the body, always remaining parallel. Hence, by Art. 128, the resultant weight of the body acts along a line which, in all positions of the body, passes through a point fixed relatively to the body. This point may thus be regarded as the point of application of the resultant weight of the body, and is called its *Centre of Gravity.*

The **Centre of Gravity** *of a rigid body is that point through which the line of action of its resultant weight passes in whatever position the body is placed.*

The distinction between the terms *Centre of Mass* and *Centre of Gravity* is important. Every body possesses a Centre of Mass, which is not necessarily a Centre of Gravity. The attraction of the earth upon a body of large size in comparison with the distances of its parts from the centre of the earth, is not necessarily equivalent to a single resultant force acting through a special point of the body. In this case the body possesses a centre of mass, which has important properties, but which is not a centre of gravity. In the present volume, however, we lose sight of the distinction between the two terms, because every body under consideration is of such a character that it possesses a centre of gravity coinciding with its centre of mass.

131. We shall confine our attention to the centre of gravity of an assemblage of particles in one plane.

A continuous body, whose thickness is inappreciable, so that it may be regarded as an aggregation of particles in one plane, is called a *lamina.*

132. Analytical Determination of Centre of Gravity. *To find the position of the Centre of Gravity of a system of particles of known masses, fixed relatively to one another in known positions in one plane.*

Let AB be a straight line fixed relatively to the particles in the plane in which they lie, and suppose that the perpendicular distances of the particles from AB are known, distances on one side of AB being reckoned positive, and on the other side negative.

Let m be the mass of one of the particles, situated at a

distance x from AB, and let $\bar{x}$ be the distance of the centre of gravity G from AB.

Let the plane containing the particles be placed in a vertical position with the line AB vertical. Then the weights of the different particles are all parallel to AB, and are proportional respectively to the masses of the particles. Their resultant is equal to their sum, and acts through G in a direction parallel to AB. Hence, taking moments about any point in AB, we have

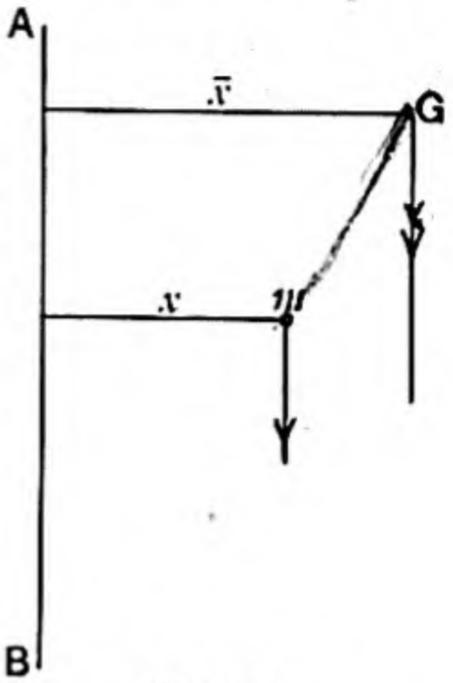


FIG. 118.

$$\{\Sigma(m)\}\bar{x} = \Sigma(mx),$$

$$\therefore\ \bar{x} = \frac{\Sigma(mx)}{\Sigma(m)}.$$

In the same way we can find the distance of G from any other line, and the knowledge of its distances from two fixed lines determines its position.

Example.—Suppose that particles of masses 3, 4, 5, 18 are situated at the angular points of a square, as indicated in Fig. 119, and that

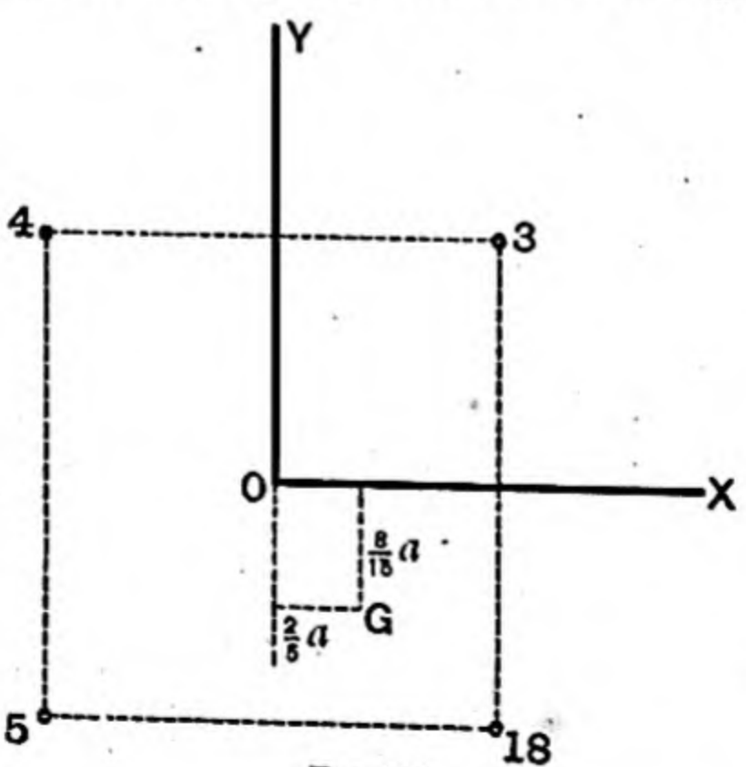


FIG. 119.

it is required to determine the distance of the centre of gravity from the centre O of the square.

Draw OX and OY parallel to the sides of the square, and let $\bar{x}$ and $\bar{y}$ be the distances of the centre of gravity from the lines OY and OX measured parallel to OX and OY respectively.

Let $2a$ be the length of a side of the square.

Then $$\bar{x}=\frac{3(a)+4(-a)+5(-a)+18(a)}{3+4+5+18}=\frac{12}{30}a=\frac{2}{5}a,$$

and $$\bar{y}=\frac{3(a)+4(a)-5(a)-18(a)}{30}=-\frac{16}{30}a=-\frac{8}{15}a.$$

Thus the centre of gravity is in the position marked G in the figure.

The distance of the centre of gravity from O

$$=\sqrt{\bar{x}^2+\bar{y}^2}$$

$$=a\sqrt{\left(\frac{2}{5}\right)^2+\left(\frac{8}{15}\right)^2}$$

$$=\frac{2a}{15}\sqrt{3^2+4^2}$$

$$=\frac{2}{3}a$$

= one-third of a side of the square.

133. Centre of Gravity of Three Equal Particles.—Let three equal particles, each of mass m, be situated at the angular points of a triangle ABC.

The resultant weight of the particles at B and C is a force equal to the weight of a mass $2m$ situated at D.

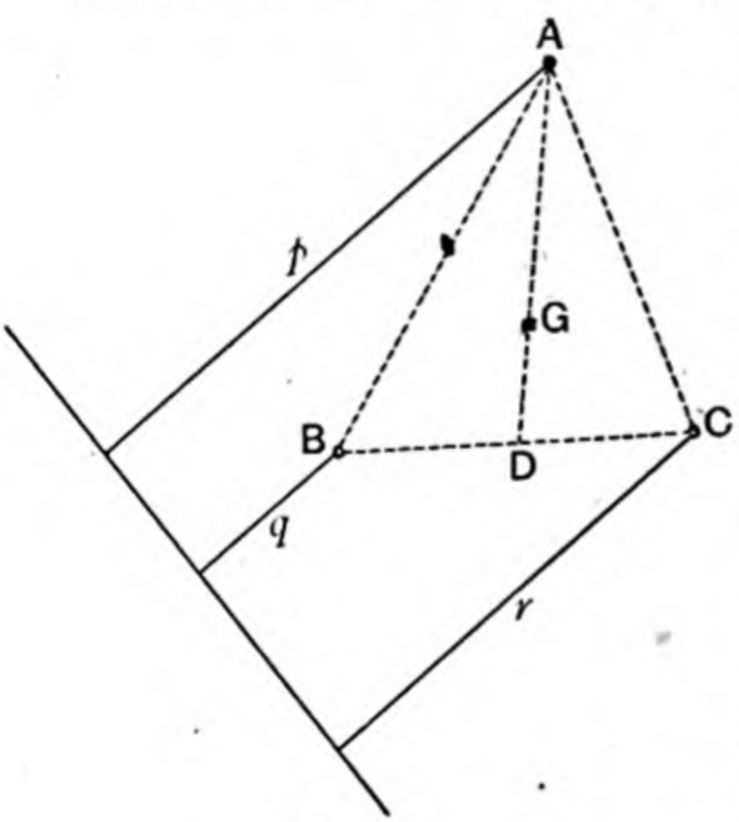


FIG. 120.

The resultant weight of $2m$ at D and m at A is a force equal to the weight of a mass $3m$ situated at G, where G divides DA in the ratio 1 : 2.

Thus the centre of gravity of the system is at G in DA, where $DG=\frac{1}{3}DA$.

We have shown that the centre of gravity G lies in the median AD. In the same way we can show that it lies in each of the other two medians of the triangle. This constitutes an independent proof of the geometrical theorem that the medians of a triangle are concurrent, and that their point of concurrence is situated at a distance of two-thirds of any median from an angular point.

If the perpendicular distances of A, B, C from any line in

the plane of the triangle are p, q, r respectively, then, by the preceding article, the perpendicular distance of G from the same line is $\frac{1}{3}(p+q+r)$.

134. Centre of Gravity of a Fine Uniform Rod.—Let AB be a fine uniform straight rod, and G its middle point.

As the rod is *uniform*, it may be regarded as a series of particles of equal mass arranged at equal indefinitely small distances apart in a straight line. These particles may be taken in pairs, G being midway between the two particles of every pair. The resultant weight of each pair acts through G, and therefore the resultant weight of the whole rod acts through G. Thus G is the centre of gravity.

We notice that if m is the mass of the rod, its resultant weight is the same as that of two masses, each of magnitude $\frac{1}{2}m$, situated at A and B.

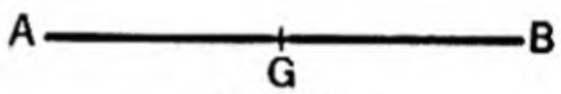


FIG. 121.

Hence, also, if p and q are the perpendicular distances of A and B from any line in the plane of the rod, then the perpendicular distance of G from the same line is $\frac{1}{2}(p+q)$.

135. Centre of Gravity of a Fine Uniform Rod bent into the shape of the Perimeter of a Triangle.—Let ABC be the triangle, and let the lengths of its sides be denoted by a, b, c as in the usual Trigonometrical notation.

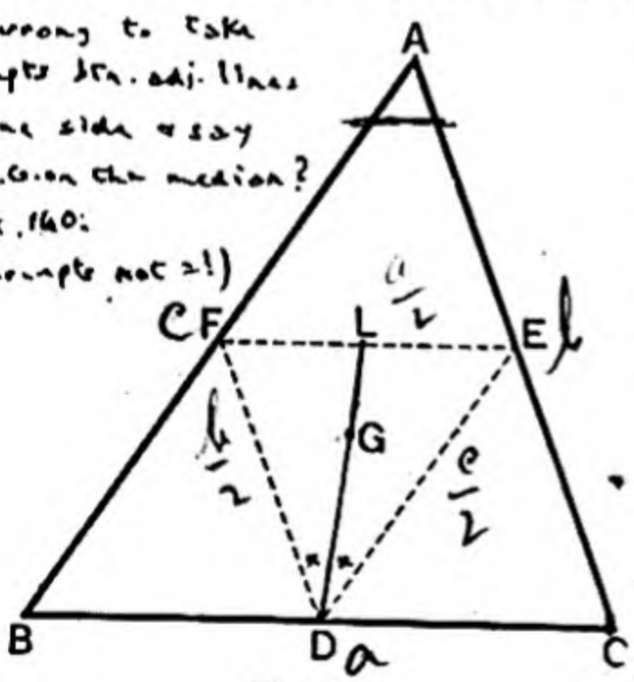


FIG. 122.

Let D, E, F be the middle points of the sides, as in the figure.

The masses of the rods are proportional to a, b, c, and their centres of gravity are at D, E, F respectively. Hence the resultant weight is the same as that of three particles, whose masses are proportional to a, b, c, situated at the points D, E, F respectively.

The resultant weight of the particles at E and F acts through L in EF, where $EL : LF = c : b$. Hence the centre of gravity of the whole system is some point G in DL.

Now the sides of the triangle DEF are respectively half as long as the sides of the triangle ABC.

$$\therefore EL : LF = ED : DF.$$

$\therefore DL$ is the bisector of the angle FDE.

Thus the centre of gravity G lies in the bisector of the angle D of the triangle DEF. Similarly it lies in the bisector of each of the other angles of the triangle DEF. Hence G coincides with the centre of the circle inscribed in the triangle DEF.

Cor. Let p, q, r be the perpendicular distances of A, B, C respectively from any line in the plane of the triangle. Then the perpendicular distances of D, E, F from the same line are $\frac{1}{2}(q+r)$, $\frac{1}{2}(r+p)$, $\frac{1}{2}(p+q)$ respectively.

Hence the perpendicular distance of G from the same line is

$$\frac{a \cdot \frac{1}{2}(q+r) + b \cdot \frac{1}{2}(r+p) + c \cdot \frac{1}{2}(p+q)}{a+b+c}$$

$$= \frac{p(b+c) + q(c+a) + r(a+b)}{2(a+b+c)}.$$

Example.—The distance of G from BC is

$$\frac{p(b+c)}{2(a+b+c)},$$

where p is the length of the perpendicular from A upon BC.

EXAMPLES XVIII

1. Particles of masses proportional to 4, 5, 8, 7 are placed in order at equal intervals along a straight line. Find the position of the centre of gravity of the system.

2. A uniform rod, of length 18 inches and mass 10 pounds, has particles of masses 11 and 6 pounds attached to its extremities. Find the centre of mass of the system.

3. A straight rod consists of two uniform parts of equal length, whose masses are as 5 to 7. Find the position of its centre of gravity.

4. A uniform rod AB, of length 20 inches and mass 6 pounds, has particles of 2, 4, 3, 5 pounds attached to it at points distant 3, 7, 12, 18 inches respectively from A. Find the position of the centre of gravity of the system.

5. In the preceding question, determine the mass of the particle which must be attached at A, in order that the centre of gravity of the whole system may be at the middle point of the rod.

6. A straight stick, of length 35 inches and mass $1\frac{1}{2}$ pounds, balances about a point distant 20 inches from one end. If a ferrule of mass one ounce is now fixed on to that end, determine the new position of the centre of gravity.

7. A straight rod $ABCD$, of uniform thickness, consists of three parts, AB, BC, CD, made of different materials and of lengths 8, 7, 5 inches respectively. If AB and BC weigh 4 and 6 pounds respectively,

find the weight of CD, in order that the centre of gravity of the whole may be at its middle point.

8. A small ring, of mass m, is capable of sliding on a rigid endless wire, of mass M, whose centre of mass is in a given position. Show how to determine the position of the centre of mass of the whole system for any given position of the ring. Investigate, also, the locus of the centre of mass, as the ring is made to slide along the wire.

9. Particles of masses 4, 3, 3 are placed at the angular points A, B, C respectively of a triangle ABC. Find the position of the centre of gravity of the system.

10. Three particles are placed at the angular points of a triangle ABC. Compare their masses, if the centre of gravity of the system divides the line joining A to the middle point of BC in the ratio 2 : 3.

11. At each vertex of a triangle is placed a particle whose mass is proportional to the length of the opposite side. Show that the centre of gravity of the system is the centre of the inscribed circle.

12. The masses of two particles at A and B are m and m' respectively. If P is any point, prove that the resultant of two forces represented by $m(AP)$ and $m'(BP)$ is represented by $(m+m')GP$, where G is the centre of mass of m and m'.

Hence show that if G is the centre of mass of m, m', m'' at any three points A, B, C respectively, then the three forces represented by $m(AG)$, $m'(BG)$, $m''(CG)$ make equilibrium.

13. G is the centre of mass of a system of particles. If one of the particles, of mass m_1, is moved through a distance a in any direction, show that G moves through a distance $\frac{m_1 a}{\Sigma(m)}$ in the same direction.

14. Particles of masses 6, 5, 3 are placed at the angular points of an equilateral triangle. Find the distance of the centre of gravity of the system from the first particle.

15. Particles of masses 11, 7, 8 are placed at the angular points of an equilateral triangle. Find the distance of the centre of gravity of the system from the first particle.

16. Particles of masses 9, 11, 6, 14 are placed at the angular points of a square, taken one way round. Determine the distance of their centre of gravity from the centre of the square.

17. Particles of masses 3, 5, 10, 6 are placed at the angular points of a square, taken one way round. Determine the distance of their centre of gravity from the centre of the square.

18. OAB is a triangle and BC perpendicular upon OA. If OA, OC, BC are of lengths a, h, k respectively, find the distance from O of the centre of gravity of particles of masses m, m_1, m_2, placed at O, A, B respectively.

19. Particles of masses p, $r+m$, q, $p+m$, r, $q+m$ are placed at the angular points of a regular hexagon, taken one way round. Show that the centre of gravity of the system coincides with the centre of the hexagon.

20. Particles of masses p, $q+m$, $r-m$, $3p+4q$, $r+m$, $q-m$ are placed at the angular points A, B, C, D, E, F respectively of a regular hexagon, taken one way round. Find the position of the centre of gravity of the system.

21. OA, OB, OC are three diverging bars of the same material and same section; prove that if their centre of gravity is at O, then the sines of the angles they make are as the squares of their lengths.

22. A uniform wire is bent into the shape of the perimeter of a right-angled triangle, the two sides that include the right angle being of length 30 and 40 inches. Find the perpendicular distances of the centre of gravity of the wire from the sides of the triangle.

23. A uniform wire is bent into the shape of the perimeter of an isosceles triangle, each of the equal sides being of length a and inclined at an angle α to the base. Find the distance of the centre of gravity of the wire from the base.

24. In the preceding example, if the wire is of mass M, find the mass of a particle which must be placed at the vertex, in order that the centre of gravity of the whole system may be midway between the vertex and the base.

25. A uniform wire is bent into the shape of the perimeter of a triangle, with its sides of length 5, 12, 13 inches. Find the distances of its centre of gravity from the two shorter sides.

136. Centre of Gravity of a Uniform Body possessing a Centre of Figure.—Whenever a rigid body of uniform density possesses a centre of figure, that centre is also the centre of gravity. For, the body may be regarded as an assemblage of particles of equal mass, which may be taken in pairs, the centre of figure lying midway between the two particles of every pair. Thus the centre of gravity of each pair is at the centre of figure, and therefore the centre of gravity of the whole body coincides with its centre of figure.

137. Centre of Gravity of a Uniform Body possessing an Axis of Symmetry.—Whenever a rigid body of uniform density possesses an axis of symmetry, its centre of gravity is situated somewhere in that axis of symmetry. For, the body may be regarded as an assemblage of particles of equal mass, which may be taken in pairs, the axis of symmetry bisecting at right angles the line joining the two particles of every pair. Thus the centre of gravity of each pair is in the axis of symmetry, and therefore the centre of gravity of the whole body is in the axis of symmetry.

138. Body consisting of two Portions.—*Given the centres of gravity of the two portions of a body, to find the centre of gravity of the whole body.*

Let a rigid body consist of two portions of masses m_1 and m_2, and let G_1 and G_2 be their centres of gravity respectively.

Then the centre of gravity of the whole body will be at G in the line G_1G_2, such that

$$G_1G : GG_2 = m_2 : m_1.$$

If x_1 and x_2 are the distances of G_1 and G_2 respectively from any line, and $\bar{x}$ the distance of G from the same line, we have,

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

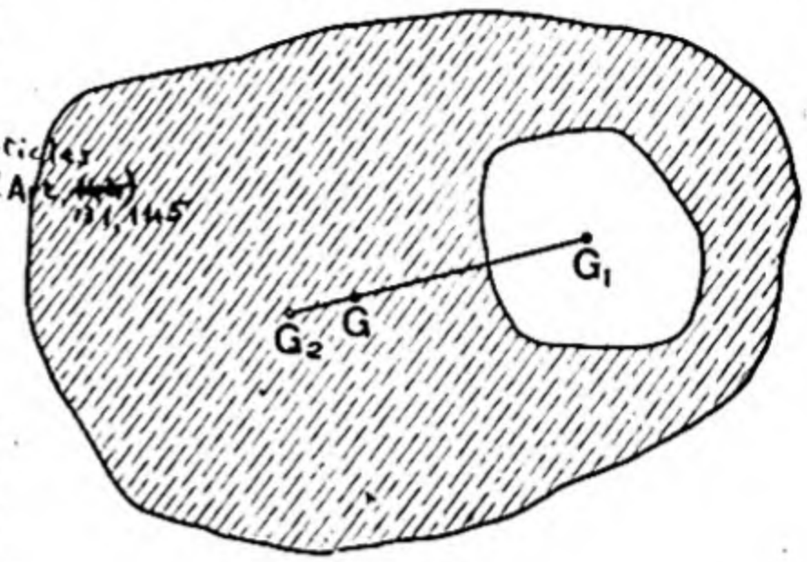


FIG. 123.

(a) In the case of a body of uniform density, the masses of the two parts will be as their *volumes*; (c) in the case of a uniform lamina, as their *areas*; in the case of a uniform rod, as their *lengths*.

139. *Given the centres of gravity of the whole of a body and of a portion of the body, to find the centre of gravity of the remainder.*

In the figure of the preceding article, we are given the positions of G and G_1, and it is required to determine the position of G_2.

Let m be the mass of the whole body and m_1 that of the portion. Then $m - m_1$ is the mass of the remainder.

The line G_1G must be produced to G_2, such that

$$G_2G : GG_1 = m_1 : m - m_1.$$

If x_1 and $\bar{x}$ are the distances of G_1 and G respectively from any line, and x_2 the distance of G_2 from the same line, we can find x_2 from the equation

$$\bar{x} = \frac{m_1 x_1 + (m - m_1) x_2}{m},$$

which gives

$$x_2 = \frac{m\bar{x} - m_1 x_1}{m - m_1}.$$

We can express this result as follows:—

The point G_2 is in the position of the centre of gravity of particles of masses m and $(-m_1)$ situated at G and G_1 respectively.

Example.—Fig. 124 represents a uniform square lamina of side 12 inches from which a rectangular portion of 8 inches by 6 inches has been removed. Let the centre of gravity G_2 of the remainder be at distances x inches and y inches from two sides of the square, as indicated.

The area of the whole square is 144 square inches.
The area of the part removed is 48 square inches.
These are as 3 : 1.

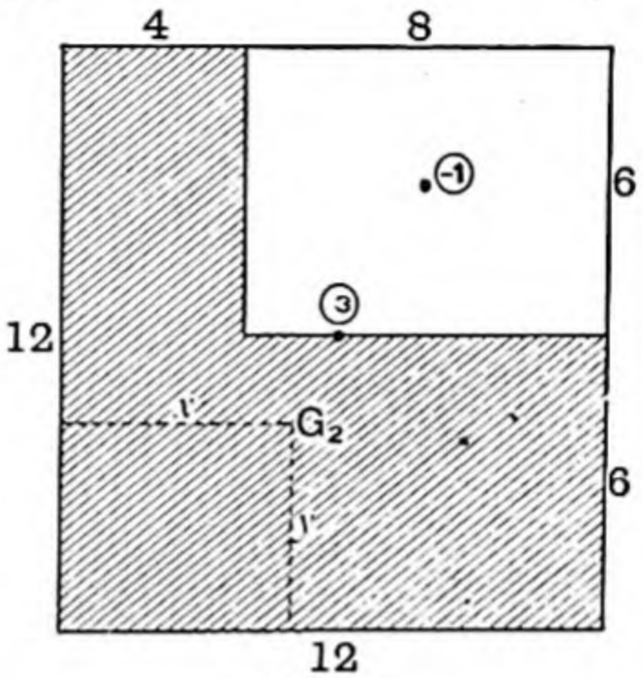


FIG. 124.

Thus we have to determine the position of the centre of gravity of particles of masses 3 and −1 situated at the centre of the square and the centre of the rectangle respectively.

Hence
$$x=\frac{(3\times 6)+(-1)8}{3-1}=5,$$
$$y=\frac{(3\times 6)+(-1)9}{3-1}=4\tfrac{1}{2}.$$

140. Centre of Gravity of a Uniform Triangular Lamina. —Let ABC be a uniform triangular lamina, and D the middle point of BC.

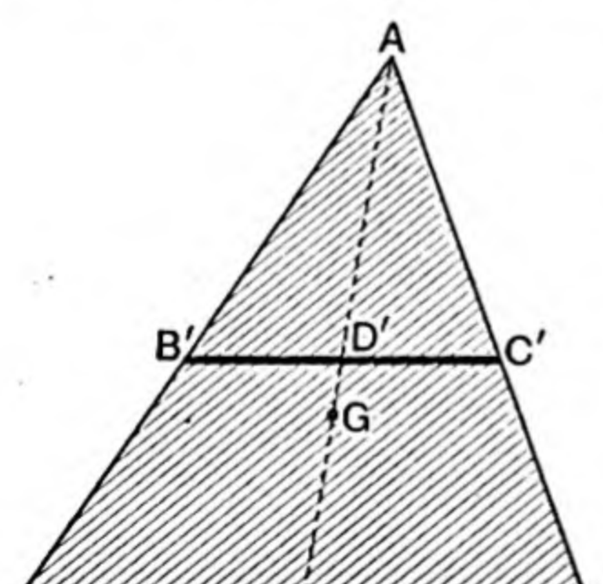


FIG. 125.

We may regard the lamina as consisting of a large number of fine rods, such as $B'C'$, all parallel to BC. As the lamina is uniform, the centre of gravity of each rod will be at its middle point. Now, by elementary geometry, the middle point D' of any such line as $B'C'$, parallel to BC, lies in the median AD. Thus the centre of gravity of each rod lies in AD. Therefore the centre of gravity of the whole lamina lies in the median AD.

In the same way it can be shown that the centre of gravity

lies in each of the other medians of the triangle. Hence the three medians must be concurrent, and the centre of gravity is at their point of concurrence.

[We have seen, in Art. 133, that the point of concurrence of the medians is at G in AD, such that $AG = 2GD$, so that the centre of gravity is at a distance of two-thirds of any median from an angular point.]

[Also, the resultant weight of the lamina is the same as that of three particles, each of one-third of the whole mass of the lamina, situated at its angular points.

[Hence also, if p, q, r are the perpendicular distances of A, B, C respectively from any line in the plane of the triangle, then the perpendicular distance of G from the same line is $\frac{1}{3}(p+q+r)$.]

141. Ex. 31.—*ABC is a uniform triangular lamina, and D is the middle point of BC. A point X is taken in AD, and the portion BXC is removed. It is required to find the centre of gravity of the remainder $ABXC$, and to determine the position of X in order that the centre of gravity of $ABXC$ may be at X.*

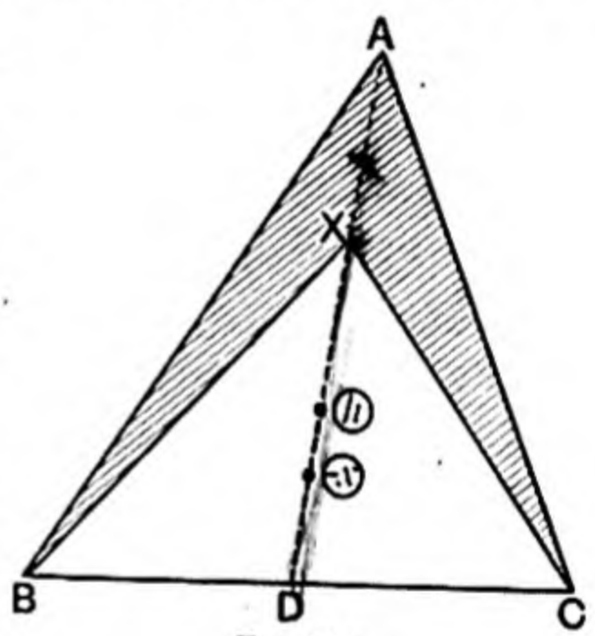


FIG. 126.

Let DA and DX be of lengths h and x respectively. Then the areas of ABC and XBC are as $h : x$. Hence we have to determine the position of the centre of gravity of particles of masses h and $-x$ situated in DA at points distant $\frac{1}{3}h$ and $\frac{1}{3}x$ respectively from D. The centre of gravity will be in DA at a distance $\bar{x}$ from D, such that

$$\bar{x} = \frac{h\left(\frac{h}{3}\right) + (-x).\frac{x}{3}}{h-x} = \frac{h+x}{3} = \frac{2}{3}(DY),$$

where Y is the middle point of XA.

If the centre of gravity is at X, we must have

$$x = \frac{h+x}{3},$$

$$\therefore\ 2x = h$$

$\therefore$ X must be the middle point of DA.

EXAMPLES XIX

1. Show that the resultant weight of a uniform lamina in the shape of a parallelogram is the same as that of four particles placed at the angular points, the mass of each particle being one-quarter of that of the lamina.

2. D is any point in the base BC of a uniform triangular lamina ABC. Show that the distance between the centres of gravity of ABD and ACD is $\frac{1}{3}BC$.

3. The sides BC, CA, AB of a uniform triangular lamina ABC are of lengths 13, 14, 15 inches respectively. Find the perpendicular distances of its centre of gravity from the sides.

4. A uniform triangular lamina ABC of weight W, lying on a smooth horizontal table, is just raised by a vertical force applied at A; find the magnitude of this force and that of the resultant pressure between the base BC and the table.

Show that if the applied force is not vertical, the base BC will not remain at rest.

5. ABC is a uniform triangular lamina of mass M, and D is the middle point of BC. Find the mass of the particle which must be attached at A, in order that the centre of gravity of the whole may be at the middle point of AD.

6. Show that the resultant weight of a uniform triangular lamina of mass M is the same as that of two particles, one of mass $\frac{2}{3}M$ situated at its circumcentre, the other of mass $\frac{1}{3}M$ situated at its orthocentre.

7. A straight line BOC is bisected at right angles by a straight line AOD, the lengths of AO and OD being 5 and 7 inches respectively. Find the distance of the centre of gravity of the figure $ABDC$ from O.

8. From a thin uniform circular plate, of radius 13 inches, is cut out a circular plate of radius 5 inches, the centre of the latter being 4 inches distant from that of the former. Find the position of the centre of gravity of the remainder.

9. From a uniform circular lamina, of radius 2 feet, is cut out a circular hole of radius 8 inches, the centre of the hole being at the middle point of a radius of the lamina. Find the distance of the centre of gravity of the remainder from the centre of the lamina.

10. In the preceding question, where must the hole be made, in order that the centre of gravity of the remainder may be on the circumference of the hole?

11. In a uniform rectangular lamina, of length 1 foot and breadth 4 inches, a circular hole of radius 1 inch is made, its centre being distant 2 inches from each of three edges of the lamina. Find the distance of the centre of gravity of the remainder from the remaining edge.

12. Two solid spheres, of radii 2 and 3 inches, and of the same material, are connected by a light rod of length 30 inches. Find the position of the centre of gravity of the whole.

13. $ABCD$ is a uniform lamina in the form of a parallelogram, whose diagonals intersect at O. If the portion AOB is removed, find the position of the centre of gravity of the remainder.

14. $ABCD$ is a uniform lamina in the form of a parallelogram, E and

F being the middle points of CB and CD respectively. If the portion CEF is removed, find the position of the centre of gravity of the remainder.

15. $ABCDE$ is a lamina of uniform thickness and density, and of such shape that $BCDE$ is a square, and $AB = AE$. If the centre of gravity of the lamina is in BE, find the ratio of AB to BC.

16. Four equilateral triangular laminæ, each of side a, but of different weights $3W$, $5W$, $5W$, $7W$, are placed with their sides in contact and with the heaviest triangle in the middle, so as to form an equilateral triangle of side $2a$. Find the position of its centre of gravity.

17. A uniform wire, 20 inches long, is bent into the shape indicated in Fig. 127. Find the distance of its centre of gravity from A.

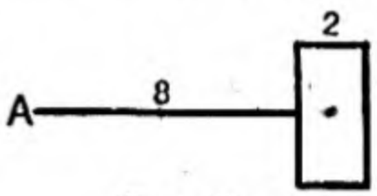


FIG. 127.

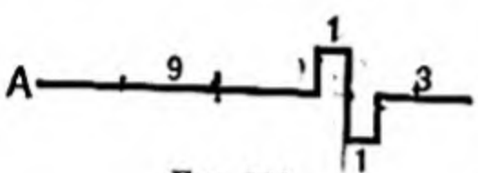


FIG. 128.

18. A uniform wire, 20 inches long, is bent into the shape indicated in Fig. 128. Find the distance of its centre of gravity from A.

19. Show that the centre of gravity of a uniform lamina, consisting of eleven equal squares as in Fig. 129, divides the line AB in the ratio 41 : 25.

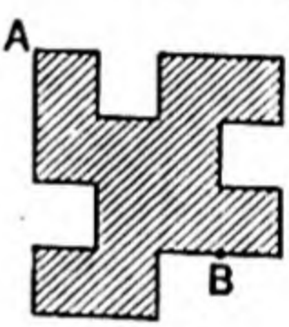


FIG. 129.

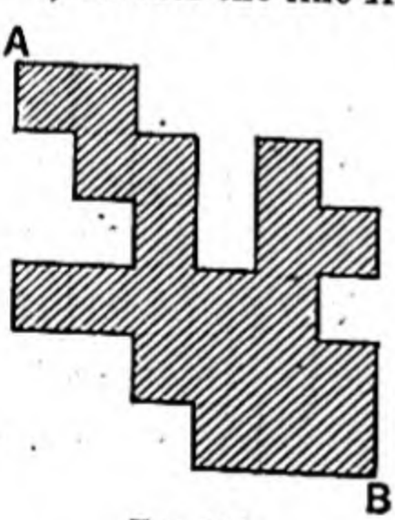


FIG. 130.

20. Show that the centre of gravity of a uniform lamina, consisting of twenty equal squares as in Fig. 130, divides the line AB in the ratio 13 : 11.

21. Find the centre of gravity of the uniform lamina represented in Fig. 131, the lengths of the different lines being given in inches.

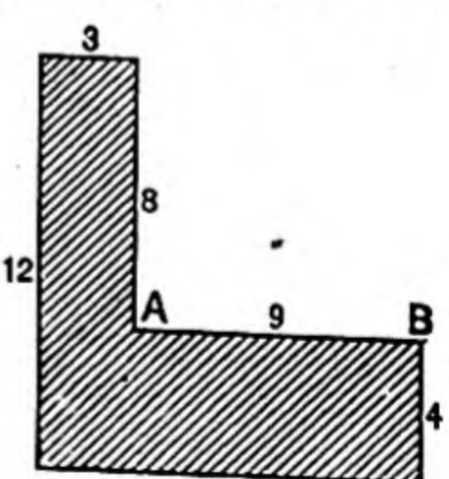


FIG. 131.

22. ABC is a triangular area, and D, E, F are the middle points of BC, CA, AB respectively. If G is the centre of gravity of the triangle ABC, and H the centre of gravity of the quadrilateral $BFEC$, find the ratio of HG to AD.

23. In the sides AB and AC of a triangle ABC, points F and E are taken, such that AF is $\frac{1}{n}$th of AB, and AE is $\frac{1}{n}$th of AC. If D is the middle point of BC, and if G and H are the centres of gravity of ABC and of the quadrilateral $BFEC$ respectively, find the ratio of HG to AD.

24. $ABCD$ and ABH are a parallelogram and a triangle on opposite sides of the same base, such that, E being the middle point of AB, HE is parallel to BC. Find the distance from E of the centre of gravity of a uniform lamina in the form $ADCBH$, the straight lines BC and HE being of lengths a and h respectively.

Find, also, the value of h, in order that the centre of gravity may be at E.

25. $ABCD$ is a uniform lamina in the form of a parallelogram, E and F being the middle points of AB and CD respectively. A point H is taken in EF, and the portion AHB is cut away. Find the distance from E of the centre of gravity of the remainder, the straight lines EF and EH being of lengths a and h respectively.

Find, also, the value of h, in order that the centre of gravity may be at H.

26. A figure is formed by taking away from a square the triangle whose angular points are the middle points of three of the sides. Find the position of its centre of gravity.

27. In the sides AB and AD of a uniform square lamina $ABCD$, points E and F are taken, such that $BE=DF=$one-quarter of a side of the square. If the portion AEF is removed, find the position of the centre of gravity of the remainder.

28. G is the centre of gravity of a uniform triangular lamina ABC; H and K are the middle points of BG and CG respectively. If the portion GHK is removed, find the position of the centre of gravity of the remainder of the lamina.

29. ABC is a uniform triangular lamina. Two triangles, each $\frac{1}{n^2}$ of the whole lamina, are removed from the corners B and C by straight lines parallel to the opposite sides. Find the position of the centre of gravity of the remainder.

30. $ABCD$ is a uniform square lamina of side a. From the corner C a square portion $CHXK$ of side x is removed. Find the value of x, in order that the centre of gravity of the remainder may be at X.

31. $ABCD$ is a square sheet of paper. The corner at A is folded over, so that A takes up the position A' in AC. Show that the centre of gravity of the sheet is moved through a distance

$$\frac{1}{6}\cdot\left(\frac{AA'}{AC}\right)^2\cdot AA'.$$

***142. Centre of Gravity of a Uniform Quadrilateral Lamina.**—Let $ABCD$ be any uniform plane quadrilateral lamina. Let H and K be the centres of gravity of the triangles ABD and CBD respectively, and let HK meet BD in F.

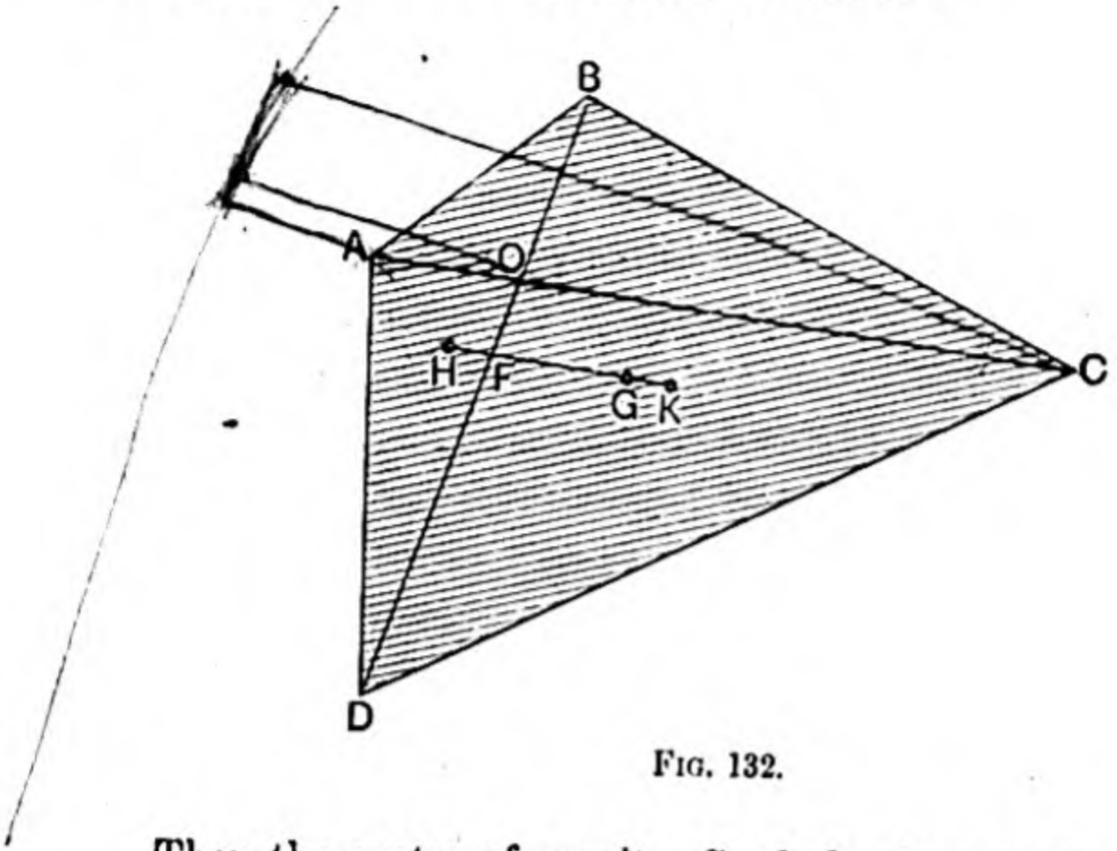


FIG. 132.

Then the centre of gravity G of the lamina lies in HK in such position that

$$\begin{aligned} HG : KG &= \text{mass of } CBD : \text{mass of } ABD \\ &= \text{area of } CBD : \text{area of } ABD \\ &= \text{perpendicular from } K \text{ on } BD \\ &\qquad : \text{perpendicular from } H \text{ on } BD \\ &= FK : HF. \end{aligned}$$

$\therefore$ componendo,

$$HK : KG = HK : HF$$
$$\therefore KG = HF.$$

This gives the position of G.

***143.** *Three times the distance of the centre of gravity of any uniform plane quadrilateral lamina from any straight line in its plane, is equal to the sum of the distances of the four corners minus that of the intersection of the diagonals from the same straight line.*

In the figure of the preceding article, let x_1, x_2, x_3, x_4 be the distances of A, B, C, D respectively from any fixed line in the plane of the paper. Then the distances of H and K from that line are $\frac{1}{3}(x_1 + x_2 + x_4)$ and $\frac{1}{3}(x_2 + x_3 + x_4)$ respectively.

Let ξ be the distance of O, the intersection of diagonals, and $\bar{x}$ the distance of G from the same fixed line.

Let h and k be the lengths of AO and OC respectively. Then, mass of ABD : mass of $CBD = h : k$.

$$\therefore \bar{x} = \frac{h \cdot \frac{1}{3}(x_1 + x_2 + x_4) + k \cdot \frac{1}{3}(x_2 + x_3 + x_4)}{h + k},$$

$$\therefore 3\bar{x} = \frac{h}{h+k}(x_1+x_2+x_3+x_4-x_3) + \frac{k}{h+k}(x_1+x_2+x_3+x_4-x_1)$$

$$= x_1+x_2+x_3+x_4 - \frac{hx_3+kx_1}{h+k}.$$

Now $$h:k = AO:OC = \xi - x_1 : x_3 - \xi$$

$$\therefore hx_3 - h\xi = k\xi - kx_1$$

$$\therefore \xi = \frac{hx_3+kx_1}{h+k}.$$

Thus $$3\bar{x} = x_1+x_2+x_3+x_4-\xi.$$

This result may be expressed as follows:—

If M is the mass of the quadrilateral, its resultant weight is the same as that of five particles of masses $\frac{1}{3}M$, $\frac{1}{3}M$, $\frac{1}{3}M$, $\frac{1}{3}M$, $-\frac{1}{3}M$ placed at A, B, C, D, O respectively.

***144.** *To find the centre of gravity of a uniform quadrilateral lamina having two parallel sides.*

This is a particular case of the preceding, but we may find the position of the centre of gravity by the following alternative method:— (non-analytical)

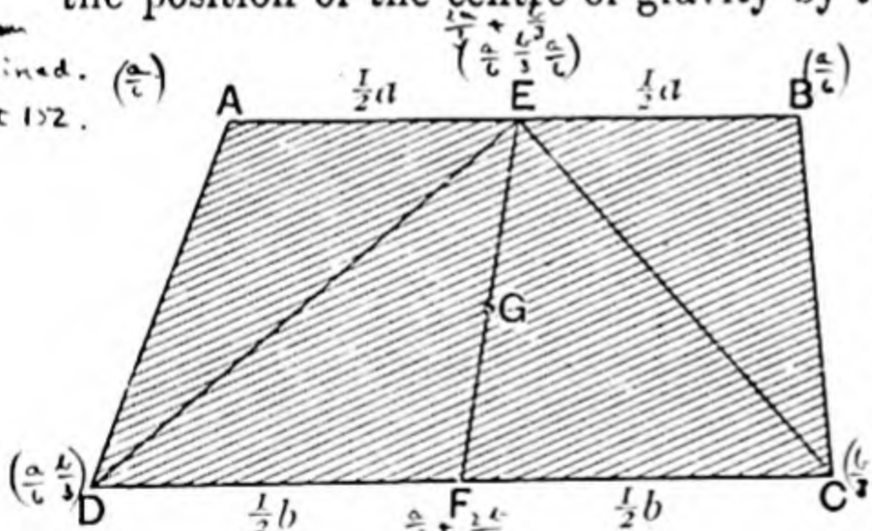


FIG. 133.

Let $ABCD$ be the lamina, having AB parallel to CD and of lengths a and b respectively. Take E and F, the middle points of AB and DC respectively. The lamina can be divided into the three triangles AED, BEC, DEC. These have the same altitude and stand on bases of lengths $\frac{1}{2}a$, $\frac{1}{2}a$, b respectively, so that their masses are proportional to $3a$, $3a$, $6b$ respectively.

Each triangle may be replaced by three particles, each of one-third of its mass, situated at its angular points; the equal particles at A and B may be replaced by one at E, and those at D and C by one at F. Thus the whole lamina may be replaced by two particles at E and F, whose masses are proportional to

$$4a+2b \text{ and } 2a+4b \text{ respectively.}$$

Hence the centre of gravity G lies in EF such that

$$EG : GF = a+2b : 2a+b.$$

145. Equilibrium of a Heavy Body Supported at One Point.—*If a heavy body is capable of turning freely about a fixed point, no forces acting upon it other than its weight and the constraint*

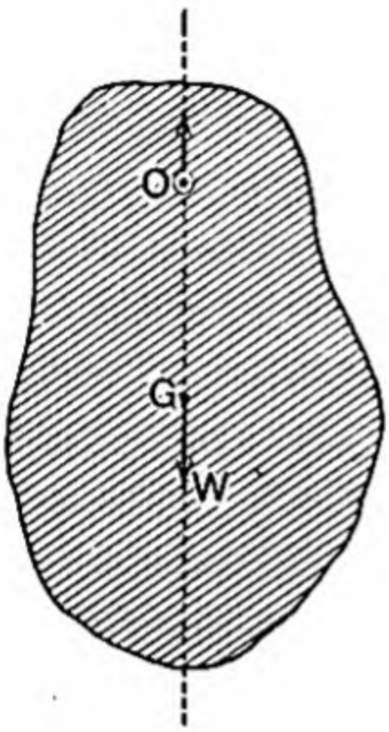


FIG. 134.

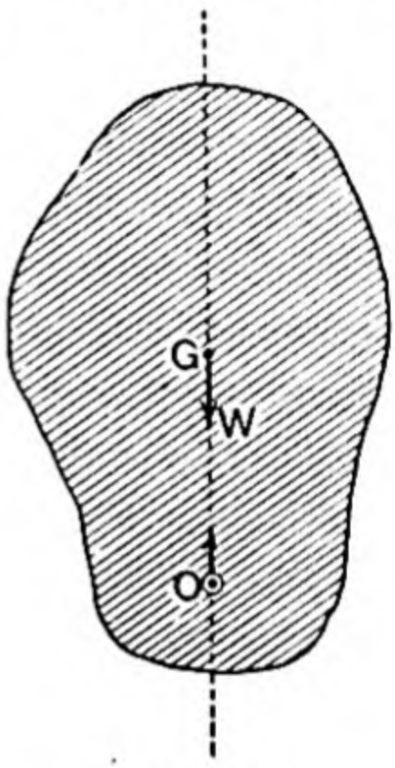


FIG. 135.

at the fixed point, then its position of equilibrium is such that its centre of gravity lies in the vertical line through the fixed point.

Let a body, of weight W, be capable of turning freely about the fixed point O, and let G be its centre of gravity.

The force of constraint at O is a self-adjusting force of such magnitude and in such direction as to keep the body, if possible, in equilibrium.

The forces which constitute the weight of the body are equivalent to a single force W acting vertically downwards through G.

These two forces must be equal and opposite and in the same straight line.

Therefore the points O and G lie in the same vertical line.

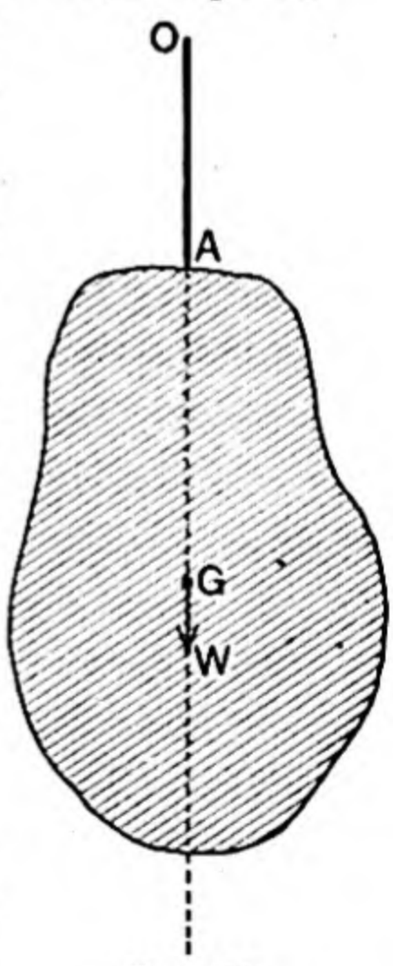


FIG. 136.

146. This apparently gives two positions of equilibrium: that in which G lies vertically below O, and that in which G lies vertically above O.

In the first position of equilibrium, with G below O, if the body is slightly displaced, the weight of the

body has a moment about O tending to restore it to its position of equilibrium. In this case the body is said to be in *Stable Equilibrium.*

In the second position of equilibrium, with G above O, if the body is slightly displaced, the weight of the body has a moment about O tending to move it still farther from its position of equilibrium. In this case the body is said to be in *Unstable Equilibrium.*

Special Case. If the body is supported by a fine string connecting the fixed point O with a point A of the body, then the only position of stable equilibrium is that in which A lies vertically below O and G vertically below A (see Fig. 136).

147. Stable, Unstable, and Neutral Equilibrium.

Def.—A body is said to be in *stable* equilibrium when, if it is slightly displaced, the forces acting upon it tend to restore it to its position of equilibrium; it is said to be in *unstable* equilibrium when, if it is slightly displaced, the forces acting upon it tend to move it still farther from its position of equilibrium; it is said to be in *neutral* equilibrium when, if it is slightly displaced, the forces acting upon it are still in equilibrium.

148. Equilibrium of One Body Resting on Another.—The general consideration of the equilibrium of one body resting upon another is beyond the scope of the present work. We shall consider only the equilibrium of a lamina resting in a vertical position upon a smooth horizontal plane, and acted upon only by its weight and the pressure of the plane. We confine our attention to the following two cases:—

1. Let the lamina rest with a straight edge AB in contact

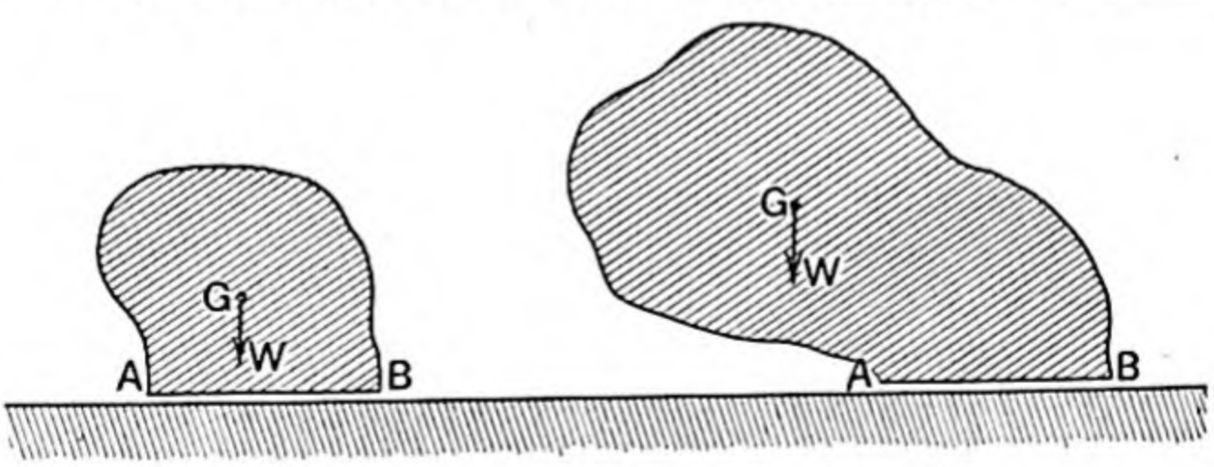


FIG. 137.

FIG. 138.

with the plane. Let G be the centre of gravity of the lamina and W its weight.

The forces which act upon the lamina are its weight, equivalent to a single force W acting vertically downwards through G, and the pressure of the plane upon the body, consisting of a number of forces acting vertically upwards at different points of AB. These latter forces are self-adjusting, and will, if possible, produce a resultant to balance W. The resultant pressure of the plane upon the body will be some force acting vertically upwards through some point between A and B.

Now, for equilibrium, the resultant pressure of the plane must be equal to W and act vertically upwards through G.

Hence, if the vertical through G meets AB between A and B, the lamina is in equilibrium; and if the vertical through G meets AB produced or BA produced, the lamina topples over.

2. Suppose that part of the boundary of the lamina is in the form of an arc of a circle, and that the lamina rests with this

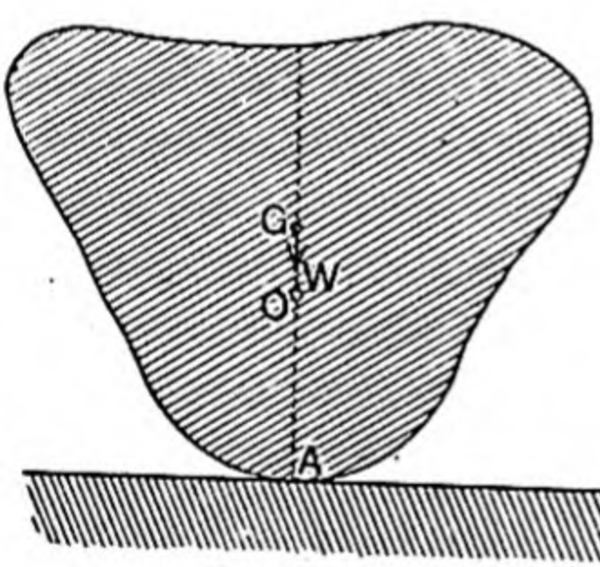


FIG. 139.

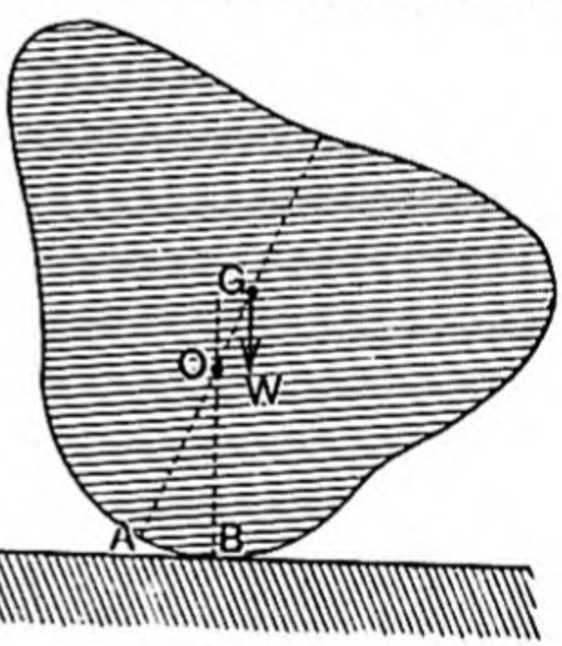


FIG. 140.

arc in contact with the horizontal plane at A. Let O be the centre of the circle, and G the centre of gravity of the lamina.

In the position of equilibrium, the vertical through A passes through both O and G. Hence the position of equilibrium is determined by placing the lamina so that that radius of the circle is vertical which passes through G.

(i) If G lies between A and O, then when the body is slightly displaced, its weight will evidently tend to restore it to its position of equilibrium.

(ii) If G lies in AO produced, then when the body is slightly displaced, its weight will evidently tend to make it move still farther from its position of equilibrium.

(iii) If G coincides with O, the body will evidently be in equilibrium in any position, so long as the arc of the circle remains in contact with the plane.

Thus the body is in stable, unstable, or neutral equilibrium according as G lies between A and O, in AO produced, or at O.

*** 149. Lamina Resting Horizontally on Three Supports.** —*A lamina, of weight W and centre of gravity G, rests in a horizontal position on three smooth supports at A, B, C. It is required to determine the pressures on the supports.*

Let P, Q, R be the forces with which the supports at A, B, C

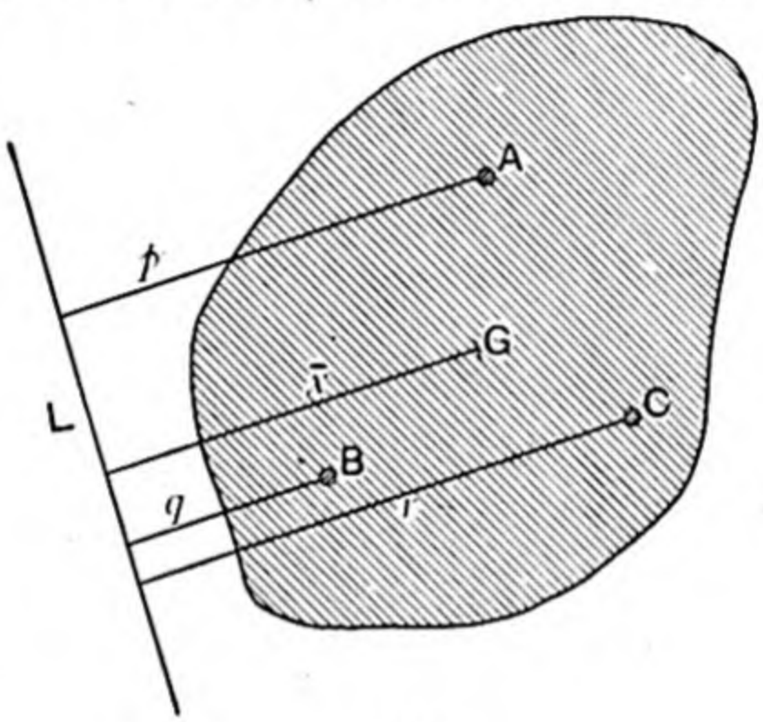


FIG. 141.

respectively press vertically upwards upon the lamina. Then the resultant of P, Q, R must be equal and opposite to W and act along the same straight line.

Thus
$$P + Q + R = W \quad . \quad . \quad . \quad \text{(i)},$$
and the centre of the parallel forces P, Q, R must coincide with G; so that if p, q, r are the perpendicular distances of A, B, C respectively from any line L in the plane of the lamina, and if $\bar{x}$ is the distance of G from the same line, we must have

$$\bar{x} = \frac{Pp + Qq + Rr}{P + Q + R} = \frac{Pp + Qq + Rr}{W};$$

$$\therefore \; Pp + Qq + Rr = W\bar{x} \quad . \quad . \quad \text{(ii)}.$$

We will describe this equation as the result of *taking moments about the line L.*

If we take moments about a second line, we obtain another equation similar to (ii).

Thus we have three simple simultaneous equations to determine the unknown quantities P, Q, R.

*** 150.** Ex. 32.—*A uniform lamina ABC, of weight W, is suspended from A. Find what vertical force must be applied at B, in order that the lamina may rest with BC horizontal.*

Suppose that the lamina rests with BC horizontal when a force P is applied vertically downwards at B.

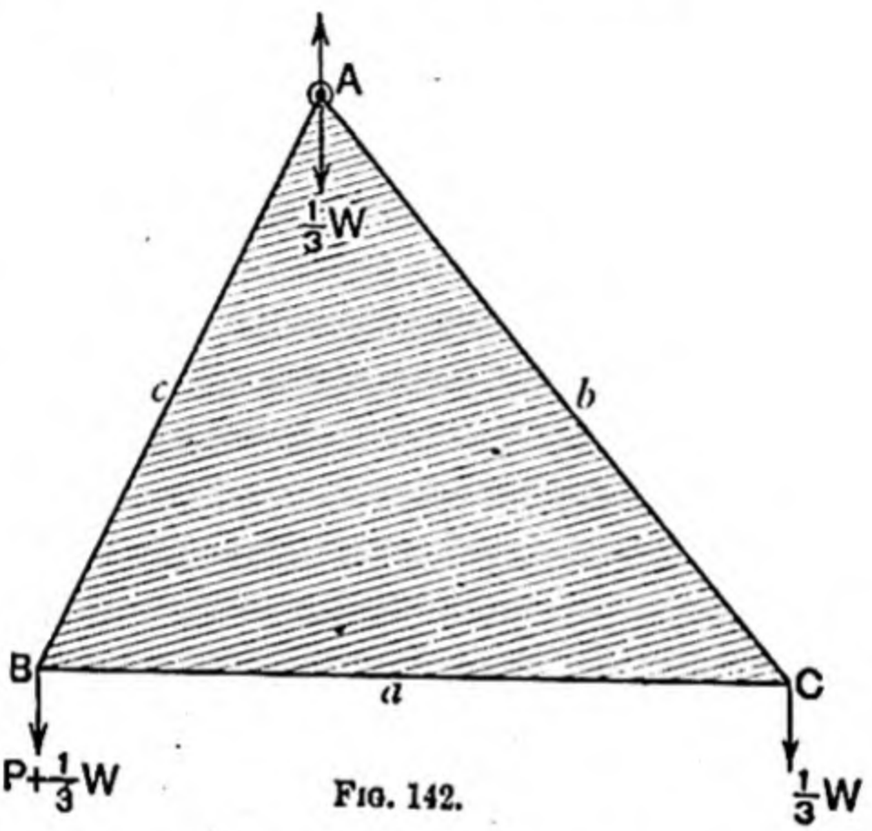


FIG. 142.

The resultant weight of the lamina is equivalent to three forces, each of magnitude $\frac{1}{3}W$, acting vertically downwards at A, B, and C.

Hence, taking moments about A, we have

$$(P+\tfrac{1}{3}W)c\cos B=\tfrac{1}{3}W.b\cos C;$$

$$\therefore\ P=\frac{W}{3}.\left(\frac{b\cos C}{c\cos B}-1\right)$$

$$=\frac{W}{3}.\left(\frac{2ab\cos C}{2ac\cos B}-1\right)$$

$$=\frac{W}{3}.\left(\frac{a^2+b^2-c^2}{a^2+c^2-b^2}-1\right)$$

$$=\frac{2(b^2-c^2)}{3(a^2+c^2-b^2)}W.$$

*** 151.** Ex. 33.—*The portion ADE of the uniform rectangular lamina $ABCD$ is removed, the straight lines DC and DE being of lengths a and b respectively. If W is the weight of the remainder,*

find the greatest weight which can be applied at A, in order that, when the lamina is placed in a vertical position with EC on a horizontal plane, it may not topple over.

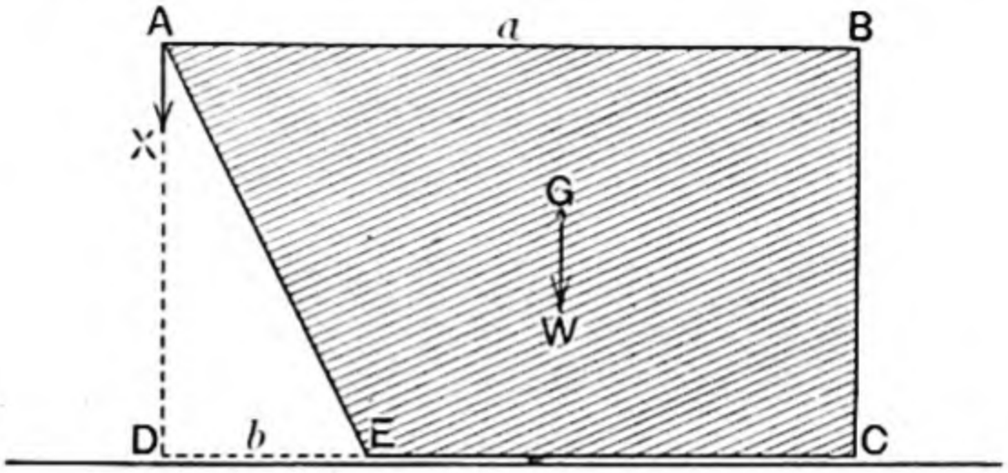


FIG. 143.

Let G be the centre of gravity of $AECB$, and x its distance from AD.

We have

$$\text{area of } ADE : \text{area of } ABCD = b : 2a.$$

Thus, the weight of ADE is $\frac{b}{2a-b}W$, and its centre of gravity is at a distance $\frac{1}{3}b$ from AD; also, the weight of $ABCD$ is $\frac{2a}{2a-b}W$, and its centre of gravity is at a distance $\frac{1}{2}a$ from AD.

Hence
$$Wx = \left(\frac{2a}{2a-b}W\right)\frac{a}{2} + \left(-\frac{b}{2a-b}W\right)\frac{b}{3} \quad \text{(Art. 139)}$$
$$= \frac{3a^2-b^2}{3(2a-b)} \cdot W.$$

Let X be the weight required. Then the resultant of X at A and W at G must pass through E.

$$\therefore (W+X)b = X(0) + Wx$$
$$= \frac{3a^2-b^2}{3(2a-b)}W.$$
$$\therefore X = \left[\frac{3a^2-b^2}{3b(2a-b)} - 1\right]W = \frac{3a^2-6ab+2b^2}{3b(2a-b)}W.$$

***152. Ex. 34.**—*The lengths of two parallel sides AB and CD of a uniform quadrilateral lamina ABCD are a and b respectively, and the weight is W. If the lamina is supported in a horizontal position at the points B, C, D, determine the pressures on the supports.*

The weights of the two triangles ABD and CBD are $\frac{a}{a+b}W$ and $\frac{b}{a+b}W$ respectively. Also the weight of each triangle is

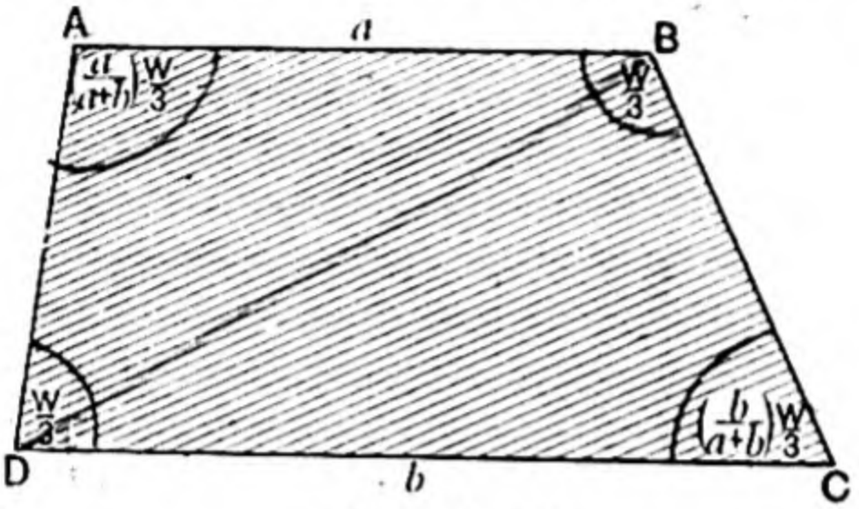


FIG. 144.

equivalent to one-third of its weight applied at each of its angular points. Thus the weight of the lamina is the resultant of $\left(\frac{a}{a+b}\right)\frac{W}{3}$ at A, $\frac{W}{3}$ at B, $\left(\frac{b}{a+b}\right)\frac{W}{3}$ at C, and $\frac{W}{3}$ at D.

Let P, Q, R be the pressures at B, C, D respectively.

Then $$P+Q+R=W.$$

Taking moments about the line CD (see Art. 149), we have

$$P=\left(\frac{a}{a+b}\right).\frac{W}{3}+\frac{W}{3}=\left(\frac{2a+b}{a+b}\right).\frac{W}{3}.$$

Taking moments about the line BC, since the perpendiculars from A and D upon BC are proportional to a and b respectively, we have

$$R.b=\left(\frac{a}{a+b}\right)\frac{W}{3}.a+\frac{W}{3}.b=\left(\frac{a^2+ab+b^2}{a+b}\right)\frac{W}{3}.$$

$$\therefore R=\frac{a^2+ab+b^2}{b(a+b)}.\frac{W}{3}.$$

Also,

$$\begin{aligned}Q&=W-P-R\\&=\frac{W}{3}\left\{3-\frac{2a+b}{a+b}-\frac{a^2+ab+b^2}{b(a+b)}\right\}\\&=\frac{W}{3}\frac{b^2-a^2}{b(a+b)}\\&=\left(\frac{b-a}{b}\right)\frac{W}{3}.\end{aligned}$$

Thus the pressures at B, C, D are

$$\frac{2a+b}{a+b}\cdot\frac{W}{3},\quad \frac{b-a}{b}\cdot\frac{W}{3},\quad \frac{a^2+ab+b^2}{b(a+b)}\cdot\frac{W}{3}$$ respectively.

* EXAMPLES XX

1. A uniform wire is bent into the form of an equilateral triangle. How must it be suspended in order that it may hang with one side vertical?

2. A uniform triangular lamina is suspended from a point of trisection of one of its sides. Show that it rests with another side vertical.

3. Why does a bicycle rider, when on the point of falling over to one side, steer his machine to that side?

4. A lamina hangs freely in a vertical plane with a point A fixed, and in this position a horizontal line BC is drawn upon it; it next hangs with B fixed and a horizontal line through A is drawn cutting BC in C; show that, if C is then fixed, the lamina will rest with the line joining A and B horizontal.

5. In a lamina of any form a line AB of length c is taken, and it is observed that when the lamina is suspended from A, the line AB dips 30° below the horizontal, and 45° when suspended from B. Find the distance of the centre of mass of the lamina from AB.

6. The length of a side of a square lamina $ABCD$ is 5 inches, but the position of its centre of gravity is unknown. It is (i) placed vertically in a smooth horizontal ring with B downwards, and with the sides BA and BC in contact with the ring; and it is observed that in the position of equilibrium (unstable) the distances of A and C from the ring are 1 and 2 inches respectively. It is then (ii) freely suspended from a point E in AD, whose distance from A is 1 inch, and it is noticed that it hangs with the diagonal BD horizontal. From these data construct geometrically the position of the centre of gravity of the lamina.

7. Two small rings A and B, of masses m and m' respectively, are fixed on a smooth circular hoop, of radius a and of no appreciable weight. If the arc AB subtends an angle 2α at the centre of the hoop, find the inclination of the straight line AB to the vertical, when the hoop is hung over a small smooth peg.

8. Two small rings A and B, of masses 6 and 16 ounces respectively, are fixed on a smooth circular hoop, of radius 7 inches and of no appreciable weight. If the straight line AB is 11 inches in length, find the inclination of AB to the vertical, when the hoop is hung over a small smooth peg.

9. A uniform wire AOB is bent at O into two straight portions inclined at an angle α, OA and OB being of lengths a and b respectively. If the portion OB is horizontal when the whole is supported at A, prove that $\cos\alpha=\dfrac{b^2}{a(a+2b)}$.

10. A uniform wire AOB is bent at O into two straight portions

inclined at angle a, OA and OB being of lengths a and b respectively. Find the inclination of OB to the horizontal, when the whole is supported at A.

11. A uniform lamina, in the shape of an isosceles triangle of vertical angle $2a$, is suspended from one extremity of the base. Show that, in the position of equilibrium, the base makes an angle $\cot^{-1}(\frac{1}{3}\cot a)$ with the horizontal.

12. A uniform wire is bent into the shape of an isosceles triangle, each of whose base angles is of magnitude a. If it is suspended from an extremity of the base, show that, in the position of equilibrium, the base is inclined at an angle $\tan^{-1}\left(2\cos a \cot\frac{a}{2}\right)$ with the horizontal.

13. Three wires, of no appreciable weight, form a triangle ABC, which is suspended from A. Particles of masses 2 pounds and 1 pound are fixed to the middle points of AB and AC respectively, and one of 3 pounds to the corner C. What must be the mass of a particle placed at B, so that in equilibrium the point of trisection of BC next B may be vertically under A?

14. A uniform lamina, in the form of a right-angled triangle ABC, is suspended from the right angle C. If $CA=2$ feet and $CB=3$ feet, find what mass must be suspended from A, in order that AB may be horizontal in the position of equilibrium, the mass of the lamina being 12 pounds.

15. A uniform triangular lamina ABC, weighing 9 pounds, is suspended from A. Find what mass must be suspended from B, in order that BC may remain horizontal, given $BC=6$, $CA=5$, $AB=4$.

16. A uniform triangular lamina ABC, of mass 18 pounds, is capable of turning freely in a vertical plane about the point A, which is fixed. If $AB=6$ inches, $BC=7$ inches, $CA=5$ inches, find what vertical force must be applied at B, so that the lamina may be supported with AB horizontal.

17. A uniform triangular lamina ABC, of weight W, is capable of turning freely in a vertical plane about the point A, which is fixed. Find what vertical force must be applied at B, so that the lamina may rest with AB horizontal.

18. A uniform triangular lamina AOB, of mass 12 pounds, having $OA=OB$ and angle $AOB=120°$, rests in a vertical position with OA on a smooth horizontal table. Find the greatest mass which can be attached at B without upsetting the lamina.

19. A uniform triangular lamina ABC just topples over when placed in a vertical position with BC upon a smooth horizontal plane. Show that $3\sin A=\sin(B\sim C)$.

20. A uniform wire is bent into the form of a triangle. If the lengths of the sides are 3, 5 and 7 inches, and the triangle is suspended from the obtuse angle, prove that, in the position of equilibrium, the shortest side is horizontal.

21. A uniform wire is bent into the form of a triangle ABC, and is suspended from the corner A. If AC is horizontal, prove the following relation between the sides of the triangle.

$$2b^2=(a-c)(a+c-b).$$

Conversely, if this relation exists between the sides of the triangle, then, when it is suspended from A, the side AC remains horizontal.

22. A uniform wire, of length $2(a+b)$ and mass M, is bent into the shape of an isosceles triangle ABC, AB and AC being the equal sides and each of length a. A particle of mass m is placed at C, and when the whole is supported at B, it rests with AC horizontal. Prove that $4b^2m=(a-2b)(a+b)M$.

23. A board of no appreciable weight, in the shape of a triangle ABC, is supported horizontally by three vertical strings attached to its corners. Find where a heavy particle must be placed on the board so that the tensions of the strings at A, B, C, may be proportional to 1, 2, 3 respectively.

24. A lamina is supported horizontally by three vertical strings. Show how to determine the tensions of the strings when a heavy sphere is placed at a given point of the lamina.

25. A uniform lamina $ABCD$ has its sides AB and DC parallel. Find the ratio of AB to DC, in order that the centre of gravity of the lamina may be in the straight line drawn through B parallel to AD.

26. The sides AB, BC, CD, DA of a uniform quadrilateral lamina are of lengths 54, 36, 27, 45 respectively, and AB is parallel to CD. Find the distance of its centre of gravity from AB.

27. The portion ADE of a uniform square lamina $ABCD$ is removed, DE being one-quarter of DC. If W is the weight of the remainder, find the greatest weight which can be applied at A, in order that, when the lamina is placed in a vertical position with EC on a smooth horizontal plane, it may not topple over.

28. The sides AB and CD of a uniform quadrilateral lamina $ABCD$ are parallel and at a distance h apart. If $AB=a$ and $CD=b$, find the distance of the centre of mass of the lamina from the straight line bisecting the opposite sides AD and BC.

29. The lengths of two parallel sides AB and CD of a uniform quadrilateral lamina $ABCD$ are a and $2a$ respectively, and the weight is W. If the lamina is supported in a horizontal position at the points B, C, D, find the pressures on the supports in terms of W.

30. A uniform plane quadrilateral $ABCD$, right-angled at A and obtuse-angled at B, is bisected by the diagonal AC. If it is placed with its plane vertical and the side AB on a horizontal plane, prove that the condition of equilibrium is $2(BM)<3(AB)$, where M is the foot of the vertical from C on AB produced.

31. Prove that the centre of mass of any uniform quadrilateral lamina may be found by the following construction: Let E and F be the middle points of the diagonals of the quadrilateral, and O their point of intersection. Along OE and OF measure $OM=\frac{2}{3}OE$ and $ON=\frac{2}{3}OF$, and complete the parallelogram $OMGN$, having OM and ON as adjacent sides. Then G will be the centre of gravity of the lamina.

32. A uniform quadrilateral lamina $ABCD$ is suspended from one corner A; M is the middle point of the diagonal BD, and O is the intersection of the diagonals. Prove that in equilibrium the vertical through A divides MO in V, so that

$$MV : VO = CO : 2(OA).$$

33. A uniform rectangular lamina $ABCD$ is bisected by a straight line cutting AB in E and CD in F. Show that the locus of the centre of mass G of either part is a parabola, and that the tangent at G to the locus is parallel to the corresponding line EF.

34. How many like coins, having diameters 20 times the thickness, can be piled on a table so that their centres may be in a straight line inclined at an angle of 45° to the horizontal?

35. A number of cubical blocks are piled one on the other so as to form a staircase, the breadth of each step being 2 inches, and the side of each block being one foot. How many may be piled up before the whole begins to topple?

36. A lamina rests in a vertical plane with a straight edge upon a vertical circle in the same plane, the circular rim being rough enough to prevent sliding. Prove that, if h is the distance of the centre of gravity of the lamina from the straight edge, and r the radius of the circle, then for stable equilibrium it is necessary that h should be less than r.

Show also that if an angle of θ radians can be found such that $h : r = \theta : \tan\theta$, then the lamina can be turned through an angle 2θ without falling.

CHAPTER IX

WORK

153. Work.—When the point of application of a force is in motion (in any direction not perpendicular to the direction in which the force is acting), the force is said to be *doing work*.

We have seen in the principle of transmissibility of force that a force may be supposed to act at any point in its line of action, so that its actual point of application may be replaced by some other point situated in the line of action of the force, the body itself remaining at rest. This is not the sort of displacement of the point of application considered in the definition of *work* given above, where we deal only with the *actual* point of application, and suppose the body on which the force acts to be in motion. The motion considered may be one which actually takes place or an imagined motion. An imagined displacement of a point is called a *virtual displacement*, and the work done by a force during an imagined displacement of its point of application is called *virtual work*.

For the present we shall consider only the case in which the force remains unaltered in direction and intensity while motion is taking place.

154. Let a force whose measure is P act at a point A in direction AX, and let the body upon which the force acts be displaced, so that the point A travels by any path into the position A', the force remaining all the time unaltered in direction and intensity.

Draw $A'M$ perpendicular upon AX, produced if necessary. Then AM is the projection of the displacement upon AX, and is positive or negative according as AM lies along AX or XA produced.

Suppose AM contains x units of length. Then the work

done by the force is measured by Px, and is positive or negative according as AM is positive or negative.

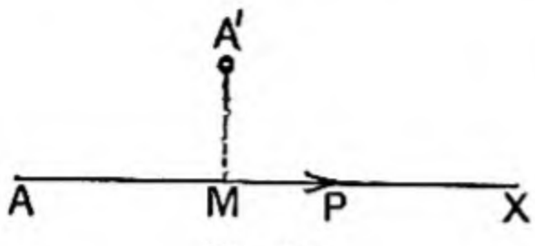


FIG. 145.

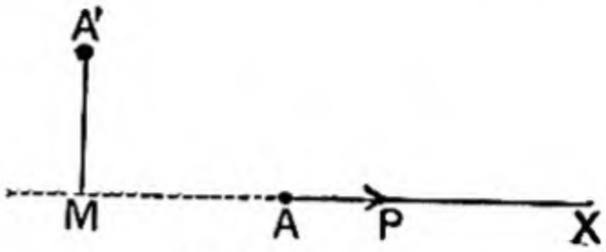


FIG. 146.

Thus, the work done by the force is given by $+Px$ or $-Px$, according as AM lies along AX or XA produced.

When the work done by a force is negative, work is said to be done *against* the force. Thus, in the second figure, the work done *by* the force is given by $-Px$, that is, the work done *against* the force is given by $+Px$.

155. The actual path of the point of application from A to A' may be of any form, for the sum of the projections of the different portions of any such path is always the same, AM.

The projection of a displacement AA' upon AX is often described as the displacement estimated in the direction AX. Thus, the work done by a force is measured by the product of the measure of the force and of the displacement of its point of application estimated in the direction of the force.

Let d be the measure of the straight line AA', and let it make an angle θ with AX. Then the work done by the force is given in magnitude and sense by

$$P \,.\, d\cos\theta$$
$$= (P\cos\theta)\,.\,d.$$

Now $P\cos\theta$ gives the resolved part of the force P estimated in the direction AA'. Thus, the work done by a force can also be measured by the product of the measure of the displacement of its point of application and that of the resolved part of the force estimated in the direction of that displacement.

156. If the point of application of a force moves in a direction perpendicular to the direction in which the force acts, no work is done by the force. For instance, if a body is displaced in a horizontal direction, no work is done by its weight; and, if a body, resting on a smooth plane, is displaced along the plane, no work is done by the resistance of the plane.

A force, however great, does no work unless its point of application moves. Thus, in its technical sense, no work is

done in an unsuccessful effort to move a body by applying a force to it.

Exs. Suppose that a body, of weight W, is at rest on a horizontal plane, and that an effort is made to move it by applying a force to it. If the force is applied vertically downwards, and the plane is strong enough to resist all tendency to penetrate it, no work is done, however great the force applied. If the force is applied vertically upwards, no work is done unless the force be increased to the value W, when any further increase, however small, will produce motion, and then work is done by the force. If the plane is perfectly smooth, then the slightest horizontal force will produce motion, and no appreciable amount of work is done by the force applied. If the plane is rough, it will resist a tendency of the body to slide over it, and no work is done until the horizontal force applied is increased to a certain value, when any further increase, however small, will produce motion, and work is then done by the force.

157. When a force undergoes alteration either in magnitude or direction or both, the consideration of the work done during any finite displacement of the body on which it acts is beyond the scope of the present volume. We assume, however, that if the displacement of the point of application is taken very small, then the ratio of the actual work done, to the work which would be done, if during this small displacement the force were to remain unaltered in direction and intensity, becomes more and more nearly a ratio of equality as we diminish the displacement; so that, when the displacement is considered indefinitely small, the actual work done and the supposed work done become ultimately equal to one another.

158. Work Done by Gravity.—*A system of particles is displaced. It is required to prove that, if the dimensions of the system and the displacements of its members are small in comparison with the distance of the system from the centre of the earth, then the work done by gravity is given by Wh, where W is the weight of the whole system and h the displacement estimated vertically downwards of the centre of mass of the system.*

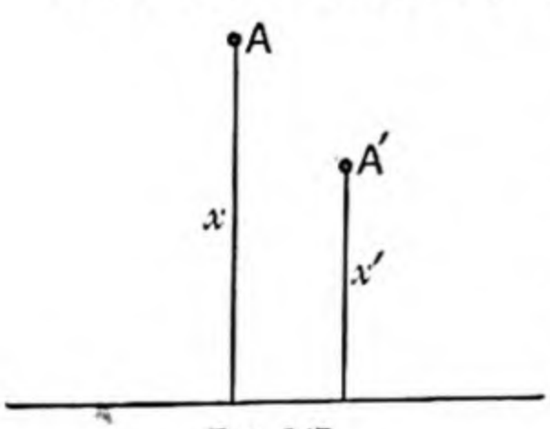


FIG. 147.

Let one of the particles, of mass m, be displaced from A to

A'. Let x be the original height of A above some fixed horizontal plane; then $\bar{x}$, the height of the centre of mass above the same plane in the first position of the system is $\frac{\Sigma(mx)}{\Sigma(m)}$. In the final position of the system, let x and $\bar{x}$ be changed into x' and $\bar{x}'$ respectively, so that $\bar{x}' = \frac{\Sigma(mx')}{\Sigma(m)}$.

Now, as the dimensions of the system are small in comparison with its distance from the centre of the earth, the weights of the different particles are in all positions of the system proportional to their masses. Also, as the displacements of the different members of the system are also small in comparison with its distance from the centre of the earth, the weight of every particle does not alter appreciably during any displacement. Hence, if w is the weight of the particle of mass m, we have

$$\bar{x} = \frac{\Sigma(mx)}{\Sigma(m)} = \frac{\Sigma(wx)}{\Sigma(w)} = \frac{\Sigma(wx)}{W},$$

and, similarly, $\bar{x}' = \frac{\Sigma(wx')}{W}$.

The work done by the weight of the particle m is

$$w(x - x') = wx - wx';$$

$\therefore$ the total work done by gravity

$$\begin{aligned} &= \Sigma(wx) - \Sigma(wx') \\ &= W\bar{x} - W\bar{x}' \\ &= W(\bar{x} - \bar{x}') \\ &= W.h. \end{aligned}$$

If the centre of mass of the system is *raised* through a vertical distance h, the work done by gravity is $-Wh$, that is, the work done *against* gravity is Wh.

The particles of the system considered are not necessarily rigidly connected together.

Example.—A uniform chain, of length a and weight W, is supported at one end, from which it hangs vertically downwards. The work done against gravity in raising the whole chain to the level of its fixed extremity is $W.\frac{1}{2}a$.

Also If, before raising the chain, a portion of length x is cut off from the lower end, then the weight of the remainder is $W.\frac{a-x}{a}$, and the work done against gravity in raising the remainder is

$$W.\frac{a-x}{a}.\frac{a-x}{2} = W.\frac{(a-x)^2}{2a}.$$

159. Work Done by Any System of Forces which Act at the Same Point.—*If a number of coplanar forces act at the same point, then the total work done by all the forces in any displacement of that point, is equal to the work which would be done by their resultant in the same displacement.*

We know that a system of forces, all acting at a point, can be replaced by a single resultant force *acting at the same point*, except when the forces are in equilibrium. The case of equilibrium can be regarded as that in which the resultant vanishes.

Now let the point of application of the forces be displaced from O to O' through a distance d.

Let P be any one of the forces acting at O, and let P_1 be its resolved part estimated in the direction OO'.

(1) Let R be the resultant force, and R_1 its resolved part in the same direction. Then we know that $R_1 = \Sigma(P_1)$.

Now the work done by the force P is equal to $P_1 d$.

Hence the total work done

$$= \{\Sigma(P_1)\}d$$
$$= R_1 d$$
$$= \text{work done by the resultant } R.$$

(2) If the forces form a system in equilibrium, the resultant R vanishes, and the work done is zero.

Thus, *If a number of coplanar forces acting at the same point of a body, are in equilibrium, no work is done in any displacement of the body, provided that the forces are unaltered in direction and intensity.*

If the forces depend in magnitude or direction or both on the position of the body, we may, in calculating the work done, consider only a very slight displacement, and the smaller the displacement, the more nearly does the method above apply.

Thus, *If a number of coplanar forces, acting at the same point of a body, are in equilibrium, no work is done in any* SLIGHT *displacement of the body.*

160. Displacement of a Rigid Body in One Plane.—Let A and B be two points of a rigid body, and P any other point of the body.

Let the body be displaced in the plane PAB, so that A, B, P come into the positions A', B', P' respectively. Then the triangles ABP and $A'B'P'$ are equal in all respects.

Draw $A'b$ parallel and equal to AB and in the same direction, and through A' and b draw $A'p$ parallel to AP and bp parallel to BP, to meet in p.

Then the triangles $A'bp$ and ABP are also equal in all respects, so that $A'p$ is equal and parallel to AP (and therefore equal to AP'), and $\widehat{bA'p} = \widehat{BAP} = \widehat{B'A'P'}$.

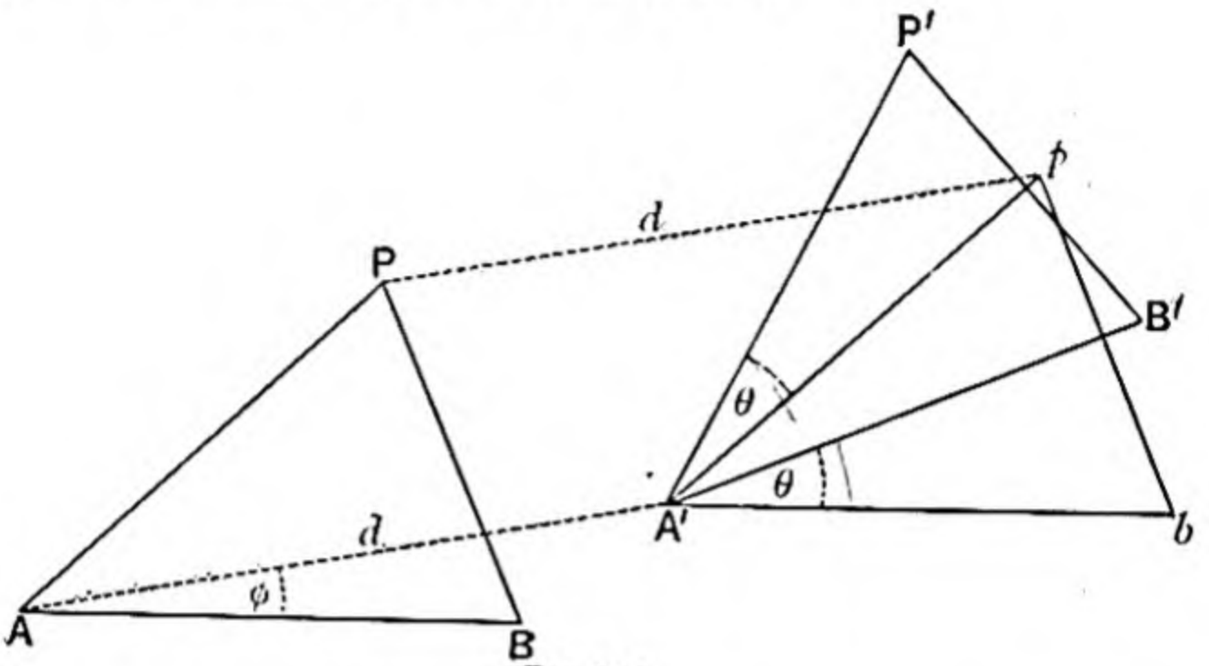


FIG. 148.

From the fact that $A'p$ is equal and parallel to AP, we deduce that Pp is equal and parallel to AA'; and from the fact that $\widehat{bA'p} = \widehat{B'A'P'}$, we deduce that $\widehat{pA'P'} = \widehat{bA'B'}$.

Let AA' be of length d in a direction making an angle ϕ with AB, and let $\widehat{bA'B'} = \theta$. Then we see that the displacement of P is equivalent to two displacements Pp and pP'; the first of these is a displacement d in a direction making an angle ϕ with AB, and the second is due to a rotation of the body about A' through an angle θ.

Now if we take different points P of the body, keeping A and B always the same, we see that the displacement of the body is equivalent to two displacements, in the first of which every point of the body is displaced through the same distance d and in the same direction ϕ, and in the second the body is rotated about one point A' through an angle θ.

161. *Def.* When the displacements of the different points of a rigid body are of the same amount and in directions parallel to one another, the displacement of the body is described as a *displacement of translation*.

The preceding article shows that any displacement in one plane of a rigid body is equivalent to a displacement of translation together with a displacement of rotation.

162. Work Done in the Displacement of a Rigid Body. —Suppose that a number of forces act at different points of a rigid body in one plane. The system is equivalent either to (1) a single force or to (2) a couple, except when the forces are (3) in equilibrium. The case of equilibrium can be regarded as included in either of the other cases, namely, in the vanishing of the single resultant force, or in the vanishing of the moment of the resultant couple.

If the system reduces to a single resultant force, let some one point of the rigid body situated in its line of action be chosen as its point of application; and if the system reduces to a resultant couple, let that couple be regarded as consisting of two equal and opposite forces of definite magnitude acting at two definite points of the body.

1. Let the body receive a displacement of *translation*, so that every point of it is displaced through the same distance d in the same direction.

Let P be any one of the forces, and let P_1 be its resolved part estimated parallel to the direction in which each point is displaced.

Then the work done by the force P is equal to $P_1 d$, and the work done by the whole system of forces is $\{\Sigma(P_1)\}d$.

(1) Now if the forces reduce to a resultant R, whose resolved part parallel to the direction of displacement is R_1, we have

$$\Sigma(P_1) = R_1,$$

so that the total work done

$$= R_1 d$$
$$= \text{work done by the resultant } R.$$

(2) If, however, the forces reduce to a couple, then $\Sigma(P_1) = 0$, so that the total work done is zero.

(3) Also, if the forces are in equilibrium, the total work done is zero.

These results are true for any finite displacement of translation, provided that the forces are not altered in magnitude or direction. If the forces depend upon the position of the body, we may still say that the above results are true for any *slight* displacement of translation.

2. Let the body receive a *slight* displacement through an angle of θ radians about a point O.

In the first position of the body, let A be the point of application of the force P, and let a be the length of OA.

The displacement of the point A is along an arc AA' of a circle of length $a\theta$. The smaller θ is made, the more nearly does AA' tend to become a straight line in a direction perpendicular to OA.

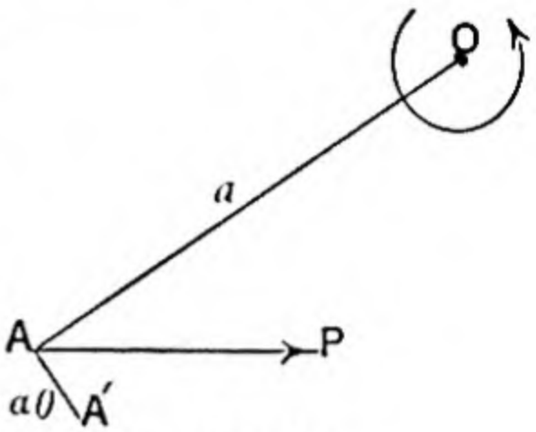


FIG. 149.

Regarding AA' then as straight and perpendicular to OA, let P_1 be the resolved part of the force P estimated in the direction AA'. Then the work done by the force P

$$= P_1 . a\theta$$
$$= \theta \times (P_1 a).$$

Now, by Art. 85, $P_1 a$ gives the moment of the force P about O estimated in the direction in which the rotation takes place.

Thus, work done by the force P

$$= \theta \times (\text{moment of } P \text{ about } O).$$

Hence, work done by the whole system of forces

$$= \theta \times (\text{sum of moments of the forces about } O).$$

(1) If the system reduces to a single resultant force, we see that in any *slight* displacement of rotation of θ radians about O, the work done is

$$\theta \times (\text{moment of resultant about } O).$$

(2) If the system reduces to a resultant couple, the work done is

$$\theta \times (\text{moment of couple}).$$

(3) If the system is in equilibrium, or if it reduces to a resultant force acting along a line through O, the work done in any *slight* displacement of rotation round O vanishes. (N.B. not $\because \theta \neq 0$)

163. Now any displacement of a rigid body in one plane has been shown to be equivalent to two displacements, one of translation and the other of rotation. Also, if a body receives two displacements, the work done by any force acting upon it is found by adding the work done in one displacement to that done in the other.

Hence, combining the above results, we see that

If any system of coplanar forces acting upon a rigid body are in equilibrium, no work is done in any SLIGHT *displacement of the body.*

164. Method of Virtual Work.—We have proved the Principle of Work for any slight displacement of a rigid body. The general consideration of the work done in a displacement of any material system is beyond the scope of the present volume. Sufficient, however, has been established to enable us to apply the principle practically to the solution of problems dealing with the equilibrium of a system of rigid bodies associated together.[1] If any slight displacement of the system is *assumed*, the *virtual* work done upon each body is zero, and therefore the total virtual work done upon the whole system is zero. The forces which act externally upon the different rigid bodies must all be noted, the work done by each force written down, and the sum equated to zero.[2] In general, a displacement may be so chosen that many of the terms vanish. For instance, if the displacement is chosen so that the point of application of one force moves in a direction perpendicular to the direction in which the force acts, no work is done by that force; also if any part of the system moves over a smooth surface, no work is done by the pressure of the surface;[4] if two bodies of the system are smoothly hinged together at one point, and if the displacement of the system is chosen so as not to break the connection between the two bodies, then the work done by the mutual forces acting on the bodies at the hinge is zero, for the action at the hinge consists of two equal and opposite forces, and the displacement of the common point of application of these two forces is the same for each force.

* **165. Work Done by the Tension of a String.**—Suppose that two bodies of a system are connected at the points A and

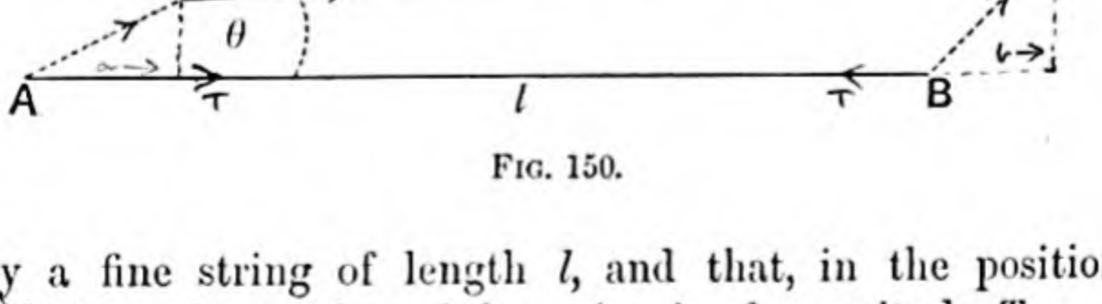


FIG. 150.

B by a fine string of length l, and that, in the position of equilibrium, the tension of the string is of magnitude T.

Let the system receive a slight virtual displacement, necessitating the cutting of the string. (We may suppose the string removed, provided that a force T is applied at A in direction AB, and an equal force T at B in direction BA.)

Let A be displaced to A' and B to B', and let l' be the length of $A'B'$ and θ the angle between $A'B'$ and AB.

Let the projections of AA' and BB' upon AB be a and b respectively. Then the work done by the two forces T is

$$Ta - Tb$$
$$= T(a - b)$$

Now the sum of the projections of AA', $A'B'$, $B'B$ upon AB is l.

i.e.
$$a + l' \cos\theta - b = l,$$
$$\therefore\ a - b = l - l' \cos\theta.$$

Now, as θ is diminished, the ratio of $\cos\theta$ to 1 becomes more and more nearly a ratio of equality.

Hence, ultimately, we have

$$a - b = l - l'.$$

Thus the work done by the two forces T in any *slight* displacement becomes ultimately equal to

$$T(l - l')$$
$$= -T \times (\text{increase in length of } AB).$$

***166.** Ex. 35.—*Two rods AB and AC, each of length a and of no appreciable weight, are smoothly jointed together at A, and rest in a vertical plane upon a smooth horizontal plane at B and C, the points B and C being connected by a fine light string. From a point D of the rod AC, at a distance b from C, there is suspended a mass of given weight W. If the rods are each inclined at an angle α to the vertical, find the tension of the string.*

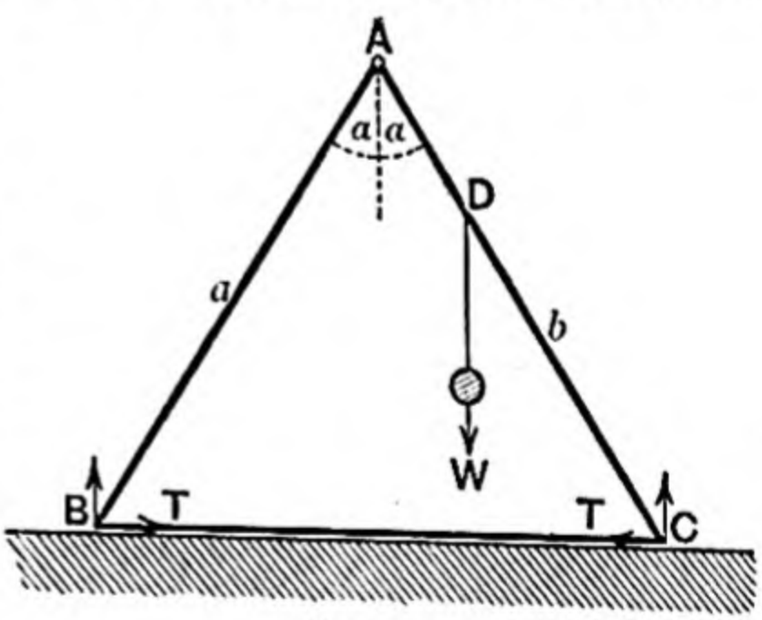


FIG. 151.

Let T be the tension of the string. Let the string be cut and a slight displacement be given to the system, such that the joint remains unbroken, while α is changed into $\alpha + \theta$.

Then the work done by the weight W is
$$= W.b[\cos\alpha - \cos(\alpha + \theta)]$$
$$= 2Wb \sin\left(\alpha + \frac{\theta}{2}\right)\sin\frac{\theta}{2};$$

and the work done by the tension of the string is

$$T \cdot 2a[\sin a - \sin(a+\theta)]$$

$$(\text{Art. 166}) = -4Ta\cos\left(a+\frac{\theta}{2}\right)\sin\frac{\theta}{2}.$$

The other ext! forces at B & C do no work.

Thus, we must have

$$Wb\sin\left(a+\frac{\theta}{2}\right) - 2Ta\cos\left(a+\frac{\theta}{2}\right) = 0.$$

Hence, making $\theta = 0$, we have

$$T = \frac{1}{2}W\frac{b}{a}\tan a.$$

***167.** Ex. 36.—*Four fine light rods are jointed together at their extremities to form a parallelogram, the opposite joints being connected by two fine light strings in a state of tension. It is required to prove that the tensions of the strings are proportional to their lengths.* cf. 205 8?

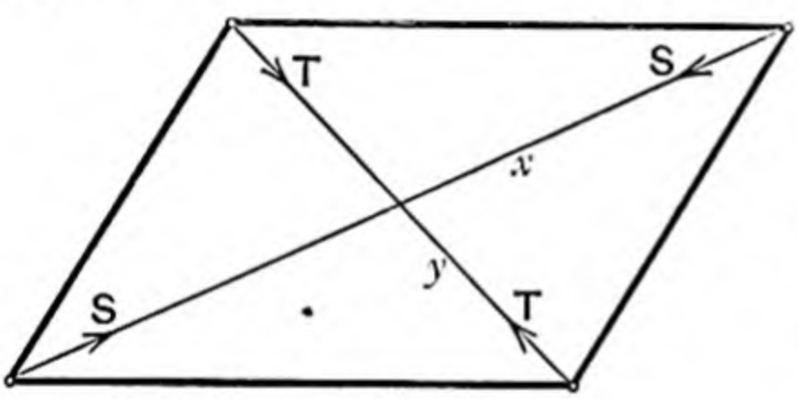


FIG. 152.

Let x and y be the lengths of the strings, and S and T their tensions.

Let the strings be cut, and such a slight virtual displacement be given to the system that the joints remain unbroken, while the lengths of the diagonals become x' and y' respectively.

Geom. Then the rods still form a parallelogram, and as the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals, we have

$$x^2 + y^2 = x'^2 + y'^2$$

$$\therefore x^2 - x'^2 = -(y^2 - y'^2)$$

$$\therefore \frac{x - x'}{y - y'} = -\frac{y + y'}{x + x'}.$$

Statics. Now, by the principle of work,

$$S(x - x') + T(y - y') = 0 \quad (\text{Art. 165})$$

N.B. Consideration of sign implicit in the principle

$$\therefore \frac{S}{T} = -\frac{y - y'}{x - x'} = +\frac{x + x'}{y + y'} = \frac{2x}{2y}, \text{ ultimately.}$$

$$\therefore S : T = x : y.$$

*EXAMPLES XXI

[Many of the Examples VIII. and XVI. can be solved by the method of virtual work.]

1. A uniform chain AB, of length a and weight W, is supported at one end A, from which it hangs vertically downwards. Find the work done against gravity in raising the lower end B through a vertical distance b, so that a loop of the chain hangs below B.

2. In the preceding question, find the work done against gravity in coiling up a length b of the chain commencing with the lower end B, so that the coil hangs vertically below A.

3. A Venetian blind consists of n bars in addition to the top fixed bar, and its weight, excluding the top bar, is W. When down, the blind is of length a, and when drawn up it is of length b. Find the work done against gravity in drawing the blind up to the top.

4. In drawing up the Venetian blind of the preceding question, the lowest bar is raised until it presses against the one above it; then the two are raised together until they press against the next; and so on. Find the work done against gravity in drawing the blind up to the position in which the rth bar from the bottom is on the point of being raised.

5. Three fine light strings are knotted together at C. One string passes round a smooth peg A and supports a mass of weight P at its free extremity; a second string passes round a smooth peg B in the same horizontal line as A, and supports a mass of weight Q; the third hangs vertically and supports a mass of weight R. Prove that the work done against gravity in carrying C from its position of equilibrium to the peg A is $2Qc\sin^2\frac{1}{2}\theta$, where c is the length of AB, and θ is the angle which BC makes with the horizontal in the position of equilibrium.

6. A right-angled triangle ABC turns stiffly in its own plane about the middle point of the hypotenuse AB. Forces just sufficient to overcome the resistance are applied thus:—P at B at right angles to BC, Q at C in direction BC, R at A at right angles to AB, all the forces tending to turn the triangle in the same sense. Show that the work done in turning the triangle through a right angle is $\frac{1}{2}\{Rc+(P-Q)(a-b)\}$, the forces remaining throughout the motion parallel to their original directions and constant in magnitude.

7. AB and BC are two uniform rods freely hinged together at B. To the ends A and C are attached small smooth rings which are free to slide on a fixed smooth horizontal rod AC. A fine light string BD, shorter than each of the rods, connects the joint B with another small smooth ring free to slide on AC. Find the tension of the string in the position of equilibrium.

8. Three equal light rods AB, BC, CD are freely jointed together at B and C, while A and D are fixed at a distance apart equal to the length of each rod. B and C are fastened by light strings to E and F situated in the same straight line with AD, such that $EA=AD=DF$. Show that the tensions of these strings are proportional to their lengths.

9. Three rhombuses $ACBD$, $CEDF$, $EGFH$, each formed of four equal light rods smoothly jointed together at their extremities, are freely connected at the extremities of their common diagonals CD and EF, the rhombuses not necessarily being in the same plane. Show that, if the diagonals AB and GH are two light strings in a state of tension, then their tensions must be proportional to their lengths.

10. Four uniform rods freely jointed together form a parallelogram $ABCD$, the weights of opposite sides being equal. If the figure is stiffened by a diagonal rod BD of no appreciable weight, and suspended freely by the corner A, then the thrust in BD is of magnitude $W\frac{BD}{AC}$, where W is half the weight of the system.

11. Four uniform rods freely jointed together form a parallelogram $ABCD$, the weights of opposite sides being equal. AB is fixed horizontally and A is attached to the opposite joint C by a light string of length l. AC is the shorter diagonal and α the acute angle of the parallelogram. Show that the tension of the string is $\frac{1}{2}W\frac{l}{a}\cot\alpha$, where W is the total weight of the four rods and a the length of the side AB.

12. A uniform square lamina rests in equilibrium in a vertical plane under gravity with two of its sides in contact with smooth pegs situated in the same horizontal line at a distance c apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by

$$c(\sin\theta+\cos\theta)=a,$$

$2a$ being the length of a side of the square.

13. A system of n equal uniform rods, smoothly jointed together at their extremities, is placed in equilibrium between two smooth parallel vertical walls, so that they rest in a vertical plane all equally inclined at an angle α to the horizontal and with successive joints in contact with different walls. Prove that, at the lowest extremity, the reaction on the rod will make with the horizontal an angle whose tangent is $\frac{2n\tan\alpha}{2n-1}$.

14. Three equal uniform rods, AB, BC, CD, are freely jointed together at B and C, and rest symmetrically, with BC horizontal, over two smooth pegs situated in the same horizontal line, AB and CD being in contact with the pegs. Prove that the inclination of AB, or CD, to the horizontal is

$$\cos^{-1}\sqrt[3]{\frac{3}{2}\cdot\frac{c-a}{a}},$$

where a is the length of each rod, and c the distance between the pegs.

CHAPTER X

MACHINES

168. Machines.—A machine is a contrivance by means of which a resistance can be overcome by a force of different intensity, or in a different direction, or along a different line to the resistance.

If a body of weight *W* were resting on the ground, an effort might be made to raise it by applying to it an upward vertical force. Here the resistance to be overcome is the weight of the body, and the force applied must be gradually increased until it is equal to *W*, and then the slightest increase beyond that value will produce motion.

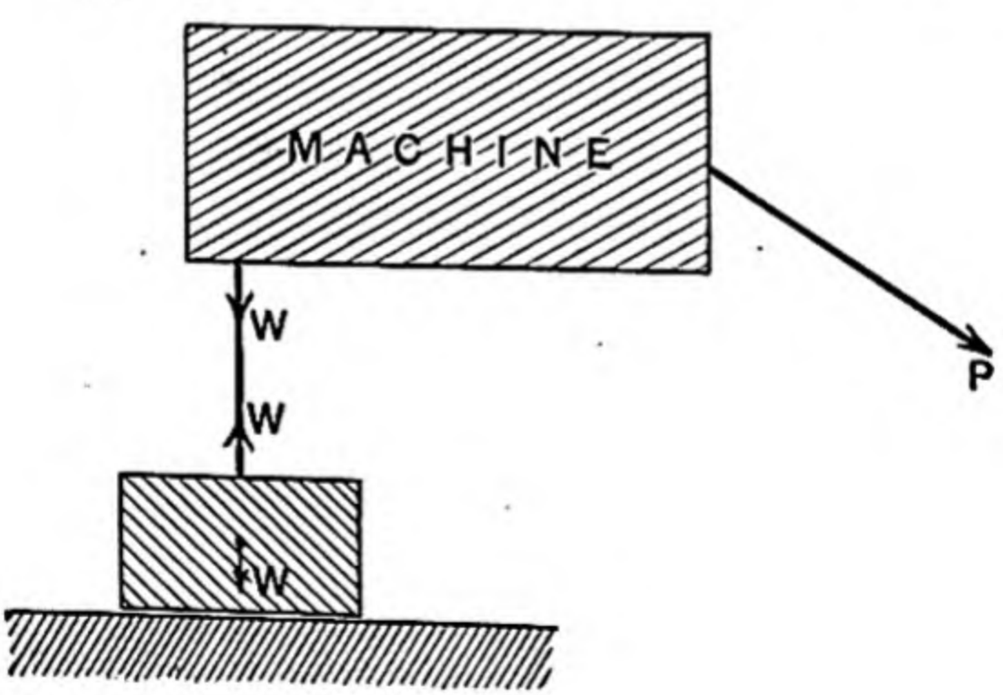


FIG. 153.

With the assistance of a machine, however, the resistance can be overcome by means of a force *P* applied at some other point.

The resistance to be overcome is called *the Weight*, and the force applied to overcome it is called *the Power*.

We confine our attention to the case in which the machine is in equilibrium. The ratio of the "Weight" to the "Power" is then called the *Mechanical Advantage* of the machine.

We shall assume that the adjustments of the different parts of a machine are all smooth. When this is the case, and when the weights of the different parts of the machine are negligible, so that the use of the machine does not introduce further resistances to be overcome, it can be proved that the following general principle holds true :—

If the "Power" and the "Weight" are adjusted so as to produce equilibrium, and if the parts of the machine are slightly displaced in a manner consistent with their geometrical connections, then the work done by the "Power" is equal to the work done against the "Weight."

We shall consider only the more simple machines, such as the *lever*, the *wheel-and-axle*, the *pulley*, the *inclined plane*, and the *screw*. The complex machines in practical use generally consist of a number of such simple parts adapted or modified to suit the purposes for which they are required.

169. The Lever.—We have considered the equilibrium of a light lever in Art. 87. With the notation of that article, the relation between the "Power" and the "Weight" is

$$Pa = Wb.$$

Thus the mechanical advantage is $\frac{a}{b}$.

If the lever is turned about the fulcrum through a small angle of θ radians, the work done by the forces P and W

$= \theta \times$ (sum of the moments of P and W about O)

$= 0.$

Thus, the work done by the "Power" is equal to that done against the "Weight."

170. If the weight of the lever is appreciable, we must include the moment of its weight in taking moments about the fulcrum. Regarding the lever in its most general form as a rigid body capable of turning freely about a fixed point, the equilibrium of a lever under any system of forces in a plane through the fulcrum has been fully considered in Art. 110.

171. The Wheel and Axle.—This apparatus is represented in Fig. 154. It consists of two cylindrical drums mounted on

the same axis. The larger drum is called the wheel and the smaller drum the axle.

The "Power" is applied by means of a rope coiled round the wheel, and the "Weight" by means of a rope coiled in the opposite direction round the axle.

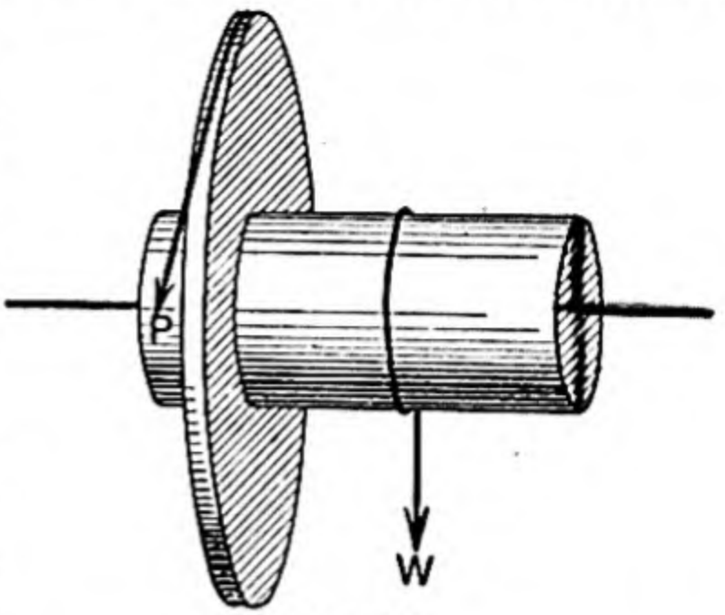


FIG. 154.

Modifications of this machine are seen in the Windlass and the Capstan. In these machines the "Power" is applied at the end of a spoke or handle. In the capstan the axis is vertical.

As the "Power" and the "Weight" are applied in different planes, the consideration of the equilibrium of the wheel and axle is properly beyond the scope of the present volume.

Fig. 155 represents a plan of the machine, the axis being perpendicular to the plane of the paper, which it intersects at O.

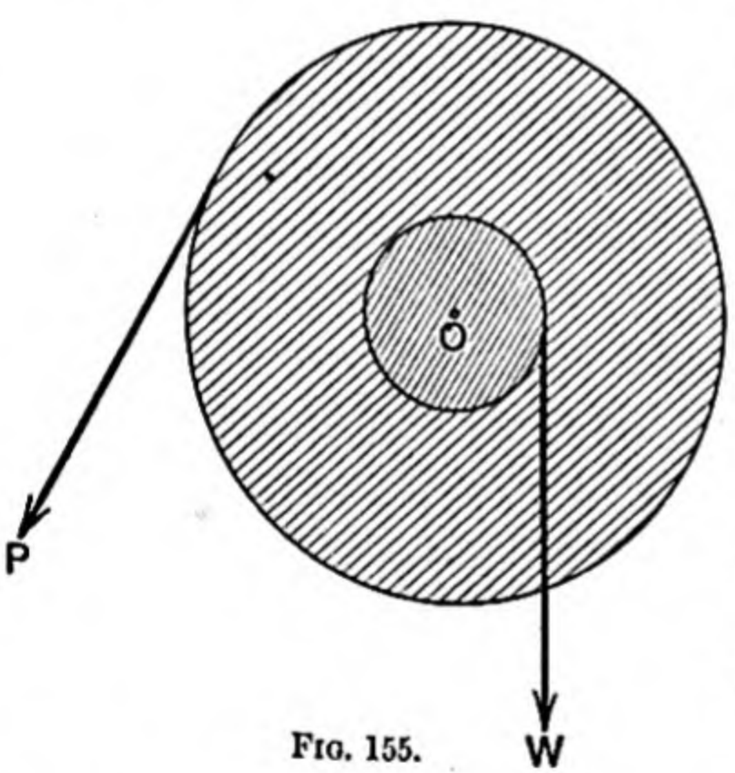


FIG. 155.

Let a be the radius of the wheel and b that of the axle. Then, regarding the apparatus as a lever with fulcrum O, we have

$$Pa = Wb,$$

so that the mechanical advantage is $\frac{a}{b}$.

If the machine is turned through an angle of θ radians in the direction in which the "Power" tends to produce rotation, a length $a\theta$ of the "Power"-rope will be uncoiled from the wheel, while a length $b\theta$ of the "Weight"-rope will be coiled on to the axle.

Thus the work done by the "Power" $= P \,.\, a\theta$, and that done against the "Weight" $= W \,.\, b\theta$.

And, as $Pa = Wb$, we see that the work done by the "Power" is equal to that done against the "Weight."

172. The Pulley.—A single fixed pulley is very useful as a machine. If the axis of the pulley is smooth, the "Power" P is equal to the "Weight" W, and the use of the pulley consists in enabling a resistance to be overcome by means of an equal force applied in a more convenient direction. The mechanical advantage is unity.

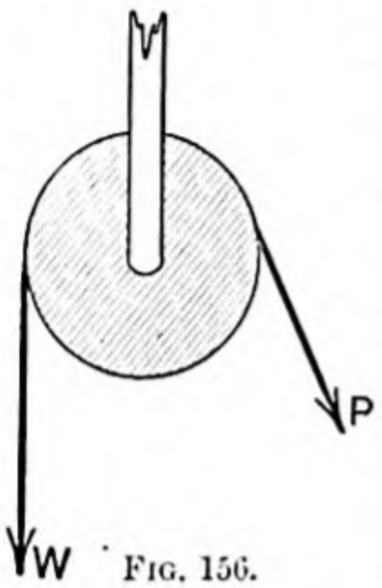


FIG. 156.

If the point of application of the "Weight" is raised through any distance, an equal length of string passes over the pulley in the direction of the "Power," so that the work done by the "Power" is equal to the work done against the "Weight."

173. We have considered the equilibrium of a single movable pulley in Art. 89. We will suppose that the string which passes round the pulley has one end fixed at the point A, the "Power" P being applied to the other end, and that the machine is used to overcome a vertical resistance W. Then, if the axis of the pulley is smooth, the string which passes round the pulley has the same tension on both sides of the pulley, and the two straight portions are equally inclined to the vertical. Also, if each portion of the "Power"-string makes an angle a with the vertical, and if the weight of the pulley and block together is of magnitude w, then

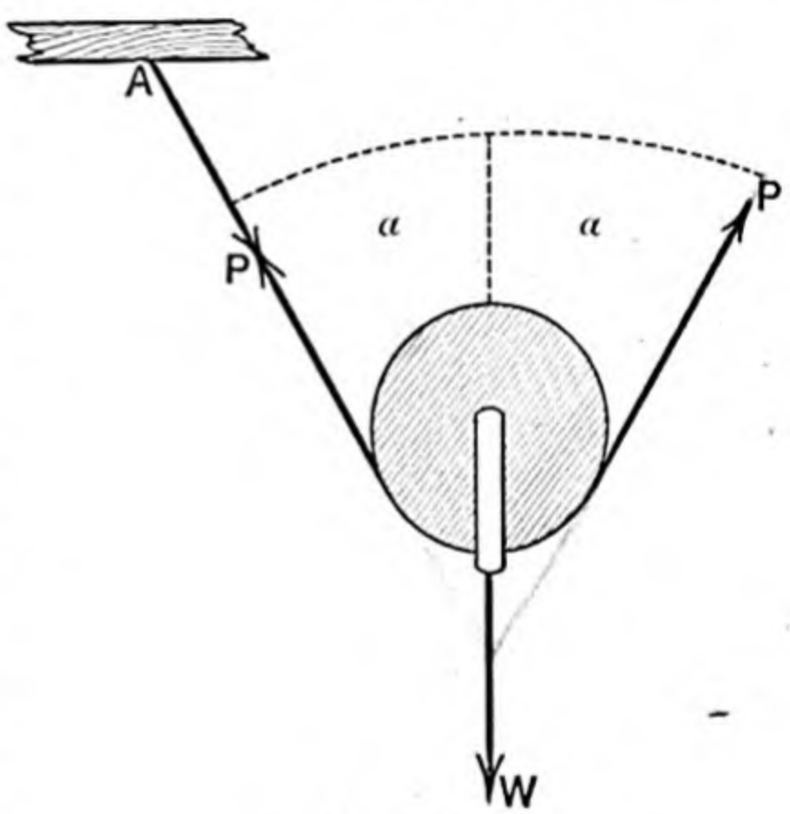


FIG. 157.

$$2P \cos a = W + w,$$

or, if the pulley is of inappreciable weight,

$$2P \cos a = W.$$

174. The most frequently occurring case is that in which all the straight portions of the string may be considered vertical without appreciable error.

In this case

$$2P = W + w,$$

or, if the pulley is of inappreciable weight,

$$2P = W,$$

so that the mechanical advantage is 2.

Let x be the distance below the level of A of the point at which the "Power" is applied, and y the distance of the centre of the pulley below the same level; also, let l be the length of the string which passes round the pulley, and c the length of the circumference of the pulley. Then, evidently, we have

$$x + l = 2y + \tfrac{1}{2}c,$$
$$\therefore\ x = 2y + \tfrac{1}{2}c - l.$$

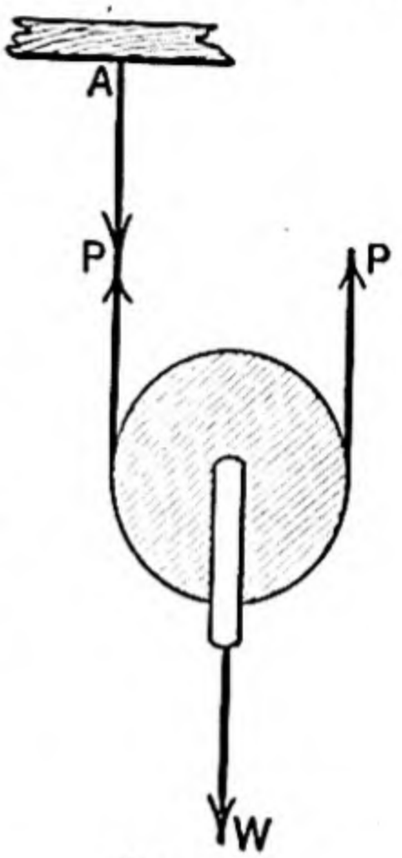


FIG. 158.

Hence, if y is diminished by any amount, x will be diminished by twice that amount. That is, if the "Weight" is raised through any distance d, then the "Power" is applied through a distance $2d$.

Thus, the work done by the "Power" is $P \,.\, 2d$,
and the work done against the "Weight" is $W \,.\, d$.
And, if the pulley is of inappreciable weight, we see that these are equal.

175. First System of Pulleys.—This is a combination of pulleys arranged in the manner indicated in Fig. 159. The string round a movable pulley is in each case attached at one end to a fixed point and at the other end to the block of the movable pulley next above it, the string round the highest movable pulley being passed over a fixed pulley to the point where the "Power" P is applied, and the "Weight" W is applied by a string to the block of the lowest pulley.

We will suppose that all straight portions of string are vertical, and that the pulleys are all of inappreciable weight.

Let the pulleys be marked $A_1, A_2, \ldots$, commencing with the lowest, and let the tensions of the strings which pass round them be $T_1, T_2, \ldots$ respectively.

Then, considering the equilibrium of each pulley separately, and supposing there are n movable pulleys in addition to the fixed pulley, we have

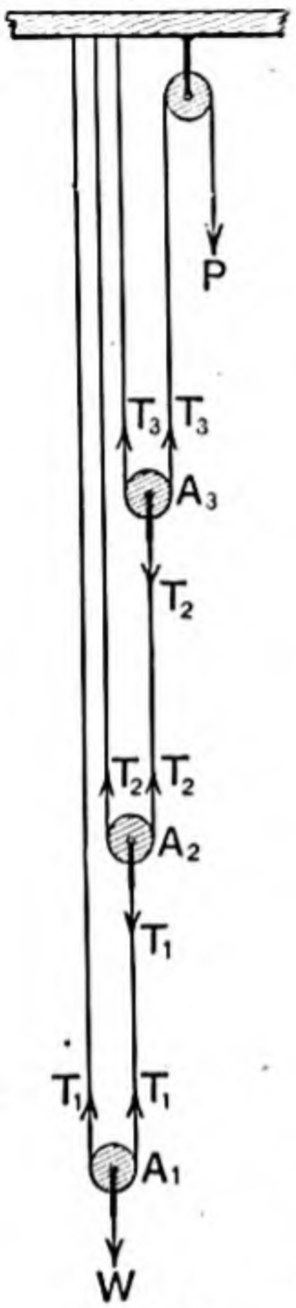


FIG. 159.

$$\begin{cases} 2T_1 = W \\ 2T_2 = T_1 \\ 2T_3 = T_2 \\ \cdots\cdots \\ 2T_n = T_{n-1} \\ T_n = P \end{cases}$$

From these equations we obtain

$$\begin{cases} T_1 = \dfrac{W}{2} \\ T_2 = \dfrac{W}{2^2} \\ T_3 = \dfrac{W}{2^3} \\ \cdots\cdots \\ P = T_n = \dfrac{W}{2^n}. \end{cases}$$

Thus, the relation between the "Power" and the "Weight" is given by the equation $W = 2^n . P$, and the mechanical advantage is 2^n.

176. In considering the work done in any displacement of the system, suppose that the point of application of the "Weight" and the pulley A_1 rise a distance x. Then, by Art. 174, we see that the end of the string which passes round this pulley rises a distance $2x$, and therefore the pulley A_2 rises a distance $2x$. Hence, similarly, the pulley A_3 rises a distance 2^2x, and so on. Thus, the pulley A_n rises a distance $2^{n-1}x$, and twice this length of string passes over the fixed pulley, so that the point of application of the "Power" descends a distance $2^n x$.

Thus, the work done by the "Power" $= 2^n x . P$,
and the work done against the "Weight" $= x . W$.
And these are equal, since $W = 2^n . P$.

177. If, in Art. 175, the pulleys $A_1, A_2, \ldots$ are of weight

w_1, w_2, . . . respectively, then, considering the equilibrium of each pulley separately, we have

$$\begin{cases} 2T_1 = W + w_1 \\ 2T_2 = T_1 + w_2 \\ 2T_3 = T_2 + w_3 \\ \cdots\cdots\cdots \\ 2T_n = T_{n-1} + w_n \\ P = T_n . \end{cases}$$

The relation between the "Power" and the "Weight" can be found by eliminating $T_1, T_2, \ldots T_n$ between these $n+1$ equations. The method of elimination is shown in the examples which follow.

If the total strain upon the supporting beam is required, we see that it has to support the weight of the whole system together with the "Power."

Thus, downward strain upon the beam

$$= P + W + w_1 + w_2 + \ldots + w_n .$$

Example 1.—Suppose there are three movable pulleys, each of weight w, and that it is required to determine what "Power" will support a mass of weight W suspended from the lowest pulley.

Referring to Fig. 159, we have

$$\begin{cases} 2T_1 = W + w \\ 2T_2 = T_1 + w \\ 2P = T_2 + w \end{cases}$$

$$\therefore \begin{cases} 2T_1 = W + w \\ 4T_2 = 2T_1 + 2w \\ 8P = 4T_2 + 4w, \end{cases}$$

$\therefore$ adding, and omitting the terms which are common to both sides, we have

$$8P = W + 7w,$$
$$\therefore P = \tfrac{1}{8}(W + 7w).$$

Example 2.—Suppose that a man of weight W hangs from the lowest pulley of the preceding example, and supports himself by pulling at the "Power" string, which passes over the fixed pulley. What must be his pull on the string?

Let T be the tension of the string between the man and the lowest pulley, then, considering the equilibrium of the man and of each pulley separately, and using the same notation as before, we have

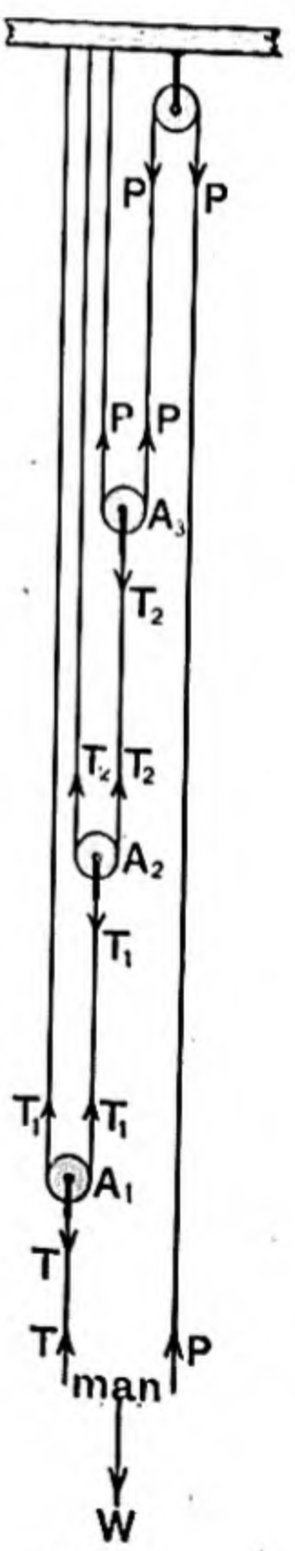


FIG. 160.

$$\begin{cases} T+P=W \\ 2T_1=T+w \\ 2T_2=T_1+w \\ 2P=T_2+w \end{cases}$$

$$\therefore \begin{cases} T+P=W \\ 2T_1=T+w \\ 4T_2=2T_1+2w \\ 8P=4T_2+4w, \end{cases}$$

$\therefore$ adding, and omitting the terms which are common to both sides, we have

$$9P=W+7w$$
$$\therefore P=\tfrac{1}{9}(W+7w).$$

178. Second System of Pulleys.—This is a combination of pulleys arranged in two blocks, one fixed and the other movable. The "Power" is applied to a string which passes over one pulley in the fixed block, then under one in the movable block, then over one in the fixed block, and so on, and is attached finally to one of the two blocks —to the fixed block if there are the same number of pulleys in use in both blocks, to the movable block if one pulley more is used in the fixed block than in the movable block. The "Weight" is applied to the movable block.

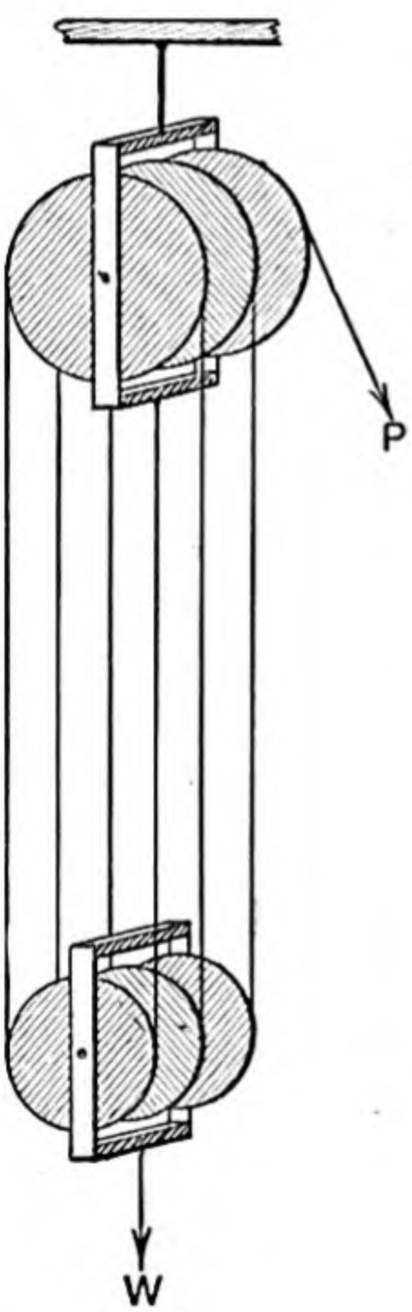


FIG. 161.

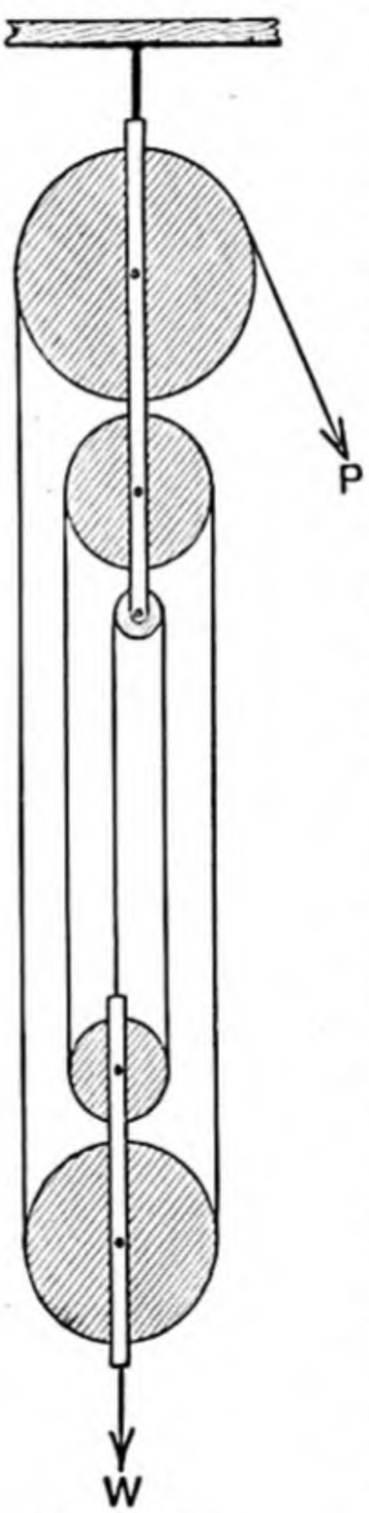


FIG. 162.

The pulleys in each block are usually arranged side by side as in Fig. 161, but they may be arranged as in Fig. 162, which is more easy to

draw, and which shows clearly the same string passing round all the pulleys.

Regarding the pulleys as smooth, the tension of the string which passes round any one pulley is the same on one side of the pulley as on the other, and as the same string passes round all the pulleys, we see that the tension of every straight portion of string is equal to the "Power" P.

We will suppose that there are n pulleys in use altogether; then there are n straight portions of string between the two blocks. Regarding all these as vertical, and considering the equilibrium of the system consisting of the movable block and the portion of string immediately adjoining it, we see that

$$nP = W + w,$$

where w is the weight of the movable block.

If the pulleys are of no appreciable weight, the relation between the "Power" and the "Weight" becomes $nP = W$.

179. Now suppose that the point of application of the "Weight" and the lower block rise through a distance x. Then each of the n portions of string has to be shortened by a distance x. Hence a length nx of string passes through the fixed block.

Thus, the work done by the "Power" $= P \cdot nx$,
and the work done against the "Weight" $= Wx$.
And these are equal, since $nP = W$.

180. Third System of Pulleys.—This is a combination of pulleys arranged in the manner indicated in Fig. 163. The topmost pulley is fixed, and all the others are movable. Each movable pulley is supported by the string which passes over the pulley next above it, and one end of each string is attached to a bar to which the "Weight" W is applied. The "Power" P is applied to the string which passes over the lowest pulley.

We will suppose that all straight portions of string are vertical, and that the pulleys and the bar are all of inappreciable weight.

Let the pulleys be marked $A_1, A_2, \ldots$, commencing with the lowest, and let the tensions of the strings which pass round them be $T_1, T_2, \ldots$ respectively. Then, considering the equilibrium of each movable pulley separately, and supposing

that there are n pulleys altogether, including the fixed pulley, we have

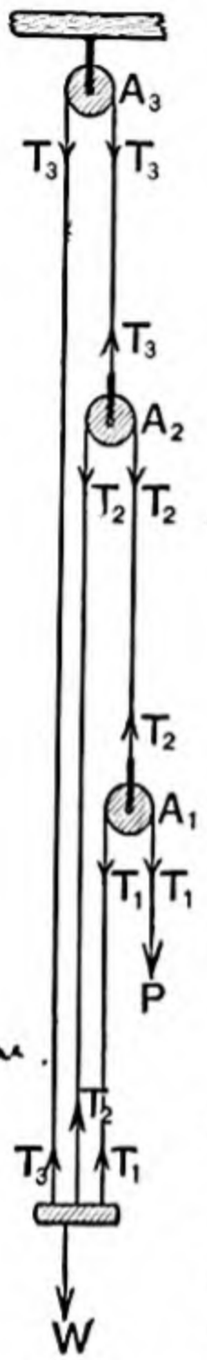


FIG. 163.

$$\begin{cases} T_1 = P \\ T_2 = 2T_1 \\ T_3 = 2T_2 \\ \cdots\cdots \\ T_n = 2T_{n-1} \end{cases}$$

Also, considering the equilibrium of the bar,

$$T_1 + T_2 + T_3 + \ldots + T_n = W$$

$$\therefore \begin{cases} T_1 = P \\ T_2 = 2P \\ T_3 = 2^2P \\ \cdots\cdots \\ T_n = 2^{n-1}P \end{cases}$$

and

$$\begin{aligned} W &= (1 + 2 + 2^2 + \ldots + 2^{n-1})P \\ &= (2^n - 1)P, \end{aligned}$$

summing the geometrical progression.

Thus, the mechanical advantage is $2^n - 1$.

181. In considering the work done in any displacement of the system, suppose that the bar and the point of application of the "Weight" rise a distance x. We will suppose that the strings which support the bar are drawn tight one after another, commencing with that which goes round the topmost pulley. We consider the displacements of the pulleys one after another, commencing this time with the topmost pulley and numbering them 1, 2, 3, . . .

In tightening the first string, a length x of it passes round the first pulley, so that the second pulley descends a distance x. This alone causes the third pulley to descend a distance $2x$. In addition to this the second string has to be drawn a further distance x round the second pulley on account of the rise in the position of the bar. Thus, altogether, the third pulley descends a distance $x + 2x$. This alone causes the fourth pulley to descend a distance $2x + 2^2x$; and also a further length x of string has to pass round the third pulley, so that altogether the fourth pulley descends a distance $x + 2x + 2^2x$.

Proceeding in this way, we see that the nth pulley from the top descends a distance $x(1 + 2 + 2^2 + \ldots + 2^{n-2})$.

This alone causes the point of application of the "Power" to descend a distance $x(2+2^2+2^3+\ldots+2^{n-1})$; and, in addition to this, a further length x of the "Power"-string has to pass round the nth pulley. Thus the point of application of the "Power" descends a distance

$$x(1+2+2^2+\ldots+2^{n-1})$$
$$=(2^n-1)x.$$

Thus, the work done by the "Power $= P(2^n-1)x$
and the work done against the "Weight" $= Wx$.
And these are equal, since $P(2^n-1)=W$.

182. If in Art. 180 the pulleys $A_1, A_2, \ldots$ are of weight $w_1, w_2, \ldots$ respectively, then, considering the equilibrium of each pulley separately, we have

$$\begin{cases} T_1 = P \\ T_2 = 2T_1 + w_1 \\ T_3 = 2T_2 + w_2 \\ \cdots\cdots\cdots \\ T_n = 2T_{n-1} + w_{n-1}. \end{cases}$$

These equations determine in order the values of $T_1, T_2, T_3, \ldots T_n$ in terms of P and $w_1, w_2, \ldots w_{n-1}$.

Also, if the bar is of weight w, we have

$$W + w = T_1 + T_2 + T_3 + \ldots + T_n.$$

Hence, by substituting for $T_1, T_2, \ldots$, we have the relation between P and W.

If it is required to determine at what point of the bar the "Weight" must be applied, we take moments about one end of the bar. Let $a_1, a_2, \ldots$ be the distances of the points of attachment of the strings from that end, a the distance of the centre of gravity of the bar from that end, and x the distance of the point of application of the "Weight" from that end. Then we have

$$Wx + wa = T_1a_1 + T_2a_2 + T_3a_3 + \ldots + T_na_n,$$

thus determining the value of x.

Example.—Suppose there are three pulleys, each of weight w and radius a, and that it is required to determine what "Power" will support a mass of weight W, and where the mass must be attached to the bar, which is of inappreciable weight.

Referring to Fig. 163, we have

$$\left\{\begin{array}{l} T_2 = 2P + w \\ T_3 = 2T_2 + w = 4P + 3w \\ \text{and } W = P + T_2 + T_3 = 7P + 4w; \end{array}\right.$$
$$\therefore\ P = \frac{1}{7}(W - 4w).$$

Let x be the distance of the point where the "Weight" is applied to the bar from the point where the "Power"-string is attached to the bar.

Then
$$\begin{aligned} Wx &= T_2 a + T_3 . 2a \\ &= (2P + w)a + (4P + 3w)2a \\ &= (10P + 7w)a \\ &= \left\{\frac{10}{7}(W - 4w) + 7w\right\}a \\ &= \frac{1}{7}(10W + 9w)a \end{aligned}$$
$$\therefore\ x = \frac{1}{7}\left\{10 + 9\left(\frac{w}{W}\right)\right\}a.$$

183. The Inclined Plane.—We have seen in Art. 69 that a particle of weight W can be supported on a smooth plane, inclined at an angle α to the horizontal, by a force P, acting at an angle θ with the plane and applied to the particle in a plane which contains the line of greatest slope of the inclined plane, provided θ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2} - \alpha$ and

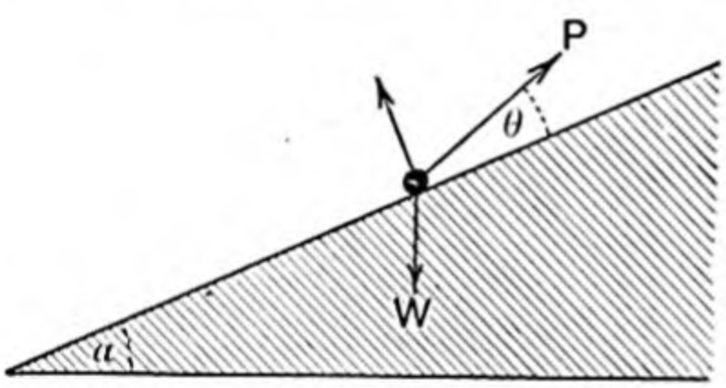


FIG. 164.

$$P\cos\theta = W\sin\alpha.$$

Regarding the inclined plane as a machine, the force W being the "Weight" and the force P the "Power," we see that the mechanical advantage is $\dfrac{\cos\theta}{\sin\alpha}$.

If the particle is displaced a distance x up the plane, the work done by the "Power" is $Px\cos\theta$, and the work done against the weight is $Wx\sin\alpha$; and these are equal, since $P\cos\theta = W\sin\alpha$.

184. Suppose that, instead of a particle, we have a rigid body of finite size, resting with a straight edge AB along a line of greatest slope of the inclined plane, and supported in this position by a force P.

Let G be the centre of gravity of the body and W its weight,

and let the force P be inclined to the plane at an angle θ. Let the normals to the plane at A and B meet the vertical through G in A' and B' respectively.

Then, resolving along the plane, we see that the relation between P and W must be the same as before; also, resolving

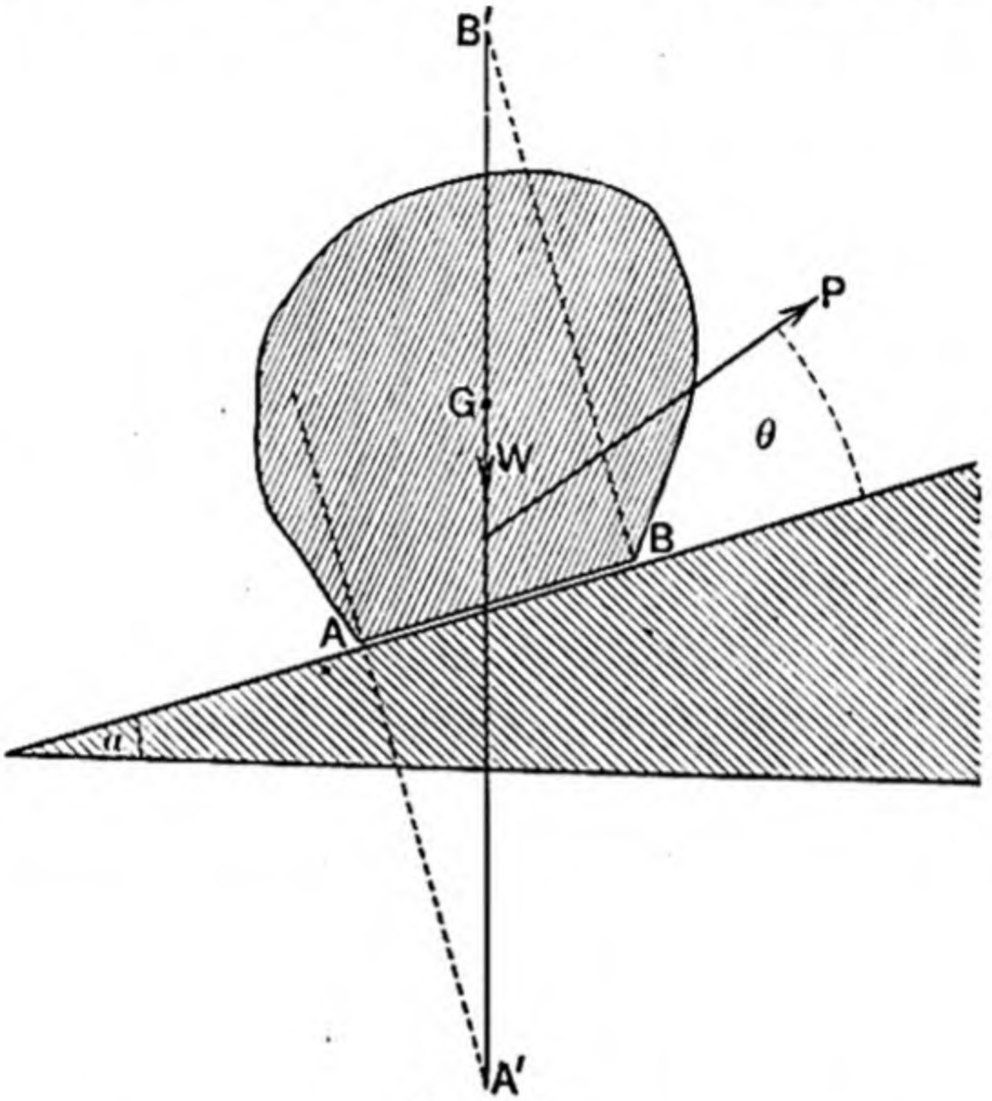


FIG. 165.

perpendicularly to the plane, we have the same limitations upon the value of θ. For equilibrium, however, it is necessary that one other condition should be satisfied:—The resultant resistance of the plane is a self-adjusting force which acts perpendicularly to the plane along a line which meets AB somewhere between A and B. It is only possible, then, for the resultant of P and W to be balanced by the resistance of the plane when the line of action of the force P intersects the vertical through G at some point between A' and B'.

Thus, for equilibrium, we must have

(i) $P\cos\theta = W\sin\alpha$;

(ii) θ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}-\alpha$;

(iii) the line of action of P must pass between A' and B'.

185. The Screw.—A *screw* consists of a right circular cylinder, round which is fixed a projecting *thread* which cuts every generating line of the cylinder at the same constant angle. The screw fits in a *nut*, containing a cylindrical hole round the inside of which a groove is cut, into which the thread of the screw exactly fits.

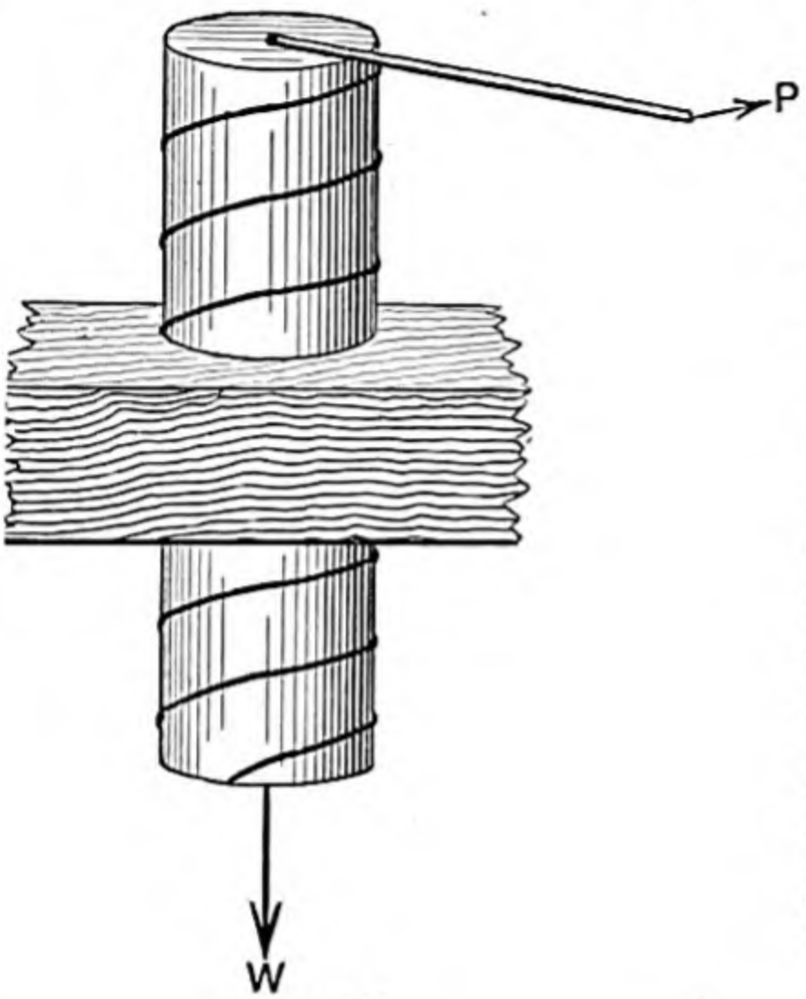


FIG. 166.

We will suppose that the screw can turn in a fixed nut with its axis vertical, and that it is used to overcome a vertical resistance W, by applying a force P in a horizontal direction to the end of an arm placed at right angles to the axis of the screw.

We will assume that the groove of the nut offers no resistance to the passage of the thread of the screw along it. This, in practice, is never even approximately true.

As the forces which act upon the screw are not in one plane, the full consideration of the equilibrium of the screw is beyond the scope of the present volume. We shall find, however, the relation between the "Power" and the "Weight" on the assumption that the Principle of Work holds good.

Let a be the length of the "Power"-arm, measured from the axis of the screw, and h the vertical distance between two consecutive threads.

Let the screw be raised through a distance h, so that it makes one revolution. Then the work done against the weight is Wh.

The point of application of the "Power" describes a curve similar in shape to the thread of the screw, and the projection of this curve upon a horizontal plane is a circle of radius a. Hence, as the "Power" always acts horizontally, the work done by it is $P \,.\, 2\pi a$. Thus, we have

$$P \cdot 2\pi a = Wh,$$

so that the mechanical advantage is $\frac{2\pi a}{h}$.

EXAMPLES XXII

[For examples on The Lever, see Sets XIII. and XVII., and for examples on The Inclined Plane, see Set IX.]

1. The radii of a wheel and axle being 1 ft. 6 in. and 3 in. respectively, find what "Power" will support a "Weight" of 60 pounds.

2. The radius of the wheel being three times that of the axle, find how far the "Weight" will be lifted when the "Power" is pulled down through a distance of 1 foot.

3. Show how to arrange three wheels and axles, each having radii R and r respectively, so that the ratio of the "Power" to the "Weight" may be $r^3 : R^3$.

4. The axle of a capstan is 9 inches in diameter, and it has six bars, each of length 5 feet measured from the axis of the capstan. With what force must each of six men push at the ends of the different bars, in order to produce a strain necessary to raise an anchor weighing 2000 pounds?

5. The radii of a wheel and axle are a and b respectively, and two masses, each of weight w, are fastened to points on the circumference of the wheel, so that the arc between them subtends an angle α at the centre. Find the position of the wheel when the greatest possible "Weight" is supported on the axle, and find what is this greatest possible weight.

6. A smooth pulley of weight w is suspended by a vertical loop of string from a wheel-and-axle and supports a mass of weight W, one end of the string being wound round the axle and the other in a contrary direction round the wheel. Find what "Power," acting at one end of an arm of length c so as to turn the axle, will keep the system in equilibrium, the radii of the wheel and axle being a and b respectively.

7. The radii of a wheel and axle are a and b respectively, and the strings which hang from them are tied to the two ends of a uniform rod of weight w, both strings being vertical and the rod horizontal. If the rod rests in this position when a mass of weight W is suspended from it, show that the point from which the mass is suspended divides the rod into two parts in the ratio

$$2bW - w(a-b) : 2aW + w(a-b).$$

8. Point out the use of a single smooth pulley (i) when fixed, (ii) when movable, and in the latter case investigate the relation between the "Power" and the "Weight" when the angle between the portions of the string diverging from the pulley is α, the weight of the pulley being inappreciable. Within what limits must α lie in order that there may be mechanical advantage?

9. A man of 10 stone stands on a platform, weighing 20 pounds, attached to a single movable pulley of mass 3 pounds, and supports

himself by pulling the free end of the string which passes over a fixed pulley, the straight portions of string being all vertical. What force must he exert?

10. What "Power" is necessary to support a mass of 150 pounds, hanging from the lower of two movable pulleys, arranged so that the string which passes round the lower pulley has one extremity fixed, while the other is attached to the upper pulley; the lower pulley weighs 8 pounds, the upper 5 pounds, and all portions of string not in contact with any pulley are vertical?

11. There are three movable pulleys, each weighing 3 pounds, arranged so that the string which passes round any pulley has one extremity fixed and the other attached to the pulley next above it. A man of 12 stone hangs from the lowest pulley and supports himself by pulling at the "Power"-string, which is passed to him over a fixed pulley. If all portions of string not in contact with any pulley are vertical, find his pull on the "Power"-string.

12. There are four movable pulleys, each weighing 2 pounds, arranged so that the string which passes round any pulley has one extremity fixed and the other attached to the pulley next above it. A man of 10 stone hangs from the lowest pulley and supports himself by pulling at the "Power"-string, which is passed to him over a fixed pulley. If all portions of string not in contact with any pulley are vertical, find his pull on the "Power"-string.

13. There are four movable pulleys, each of weight w, arranged as in the preceding question. Find what "Power" will support a mass of weight W hung from the lowest pulley.

14. There are four movable pulleys, each of weight w, arranged as in question 12. A man of weight W hangs from the lowest pulley and supports himself as before. Find his pull on the "Power"-string.

15. Find the relation between the "Power" P and the "Weight" W in the first system of pulleys, when there are n movable pulleys of weights $w_1, w_2, \ldots . w_n$, commencing with the lowest, all the straight portions of string being vertical.

16. Find the relation between the "Power" P and the "Weight" W in the first system of pulleys, when there are n movable pulleys each of weight w, all the straight portions of string being vertical.

17. In the preceding question, if the "Power" is exerted by a man of weight W standing upon the ground, find his pressure on the ground.

18. In question 16, prove that the magnitude of the resultant pull on the fixed horizontal beam, from which the pulleys are supported, is $(P-w)(2^n-1)+nw$.

In particular, if there are five pulleys, prove that the distance of the line of action of this resultant pull from the string of greatest tension is $\dfrac{26P-16w}{31P-26w}a$, where a is the radius of each pulley.

19. A system of pulleys, of no appreciable weight, are arranged in two blocks, one fixed and the other movable. Draw diagrams of such a system in which masses of (i) 60 pounds, (ii) 70 pounds are in turn supported by a force of 10 pounds' weight.

20. In a system of pulleys arranged in two blocks, one fixed and

the other movable, the same string passes round all the pulleys, and all straight portions of string are vertical. A "Power" of 226 pounds' weight will just raise a mass of $\frac{1}{2}$ ton, and a "Power" of 450 pounds' weight will just raise a mass of 1 ton. Find the mass of the lower block and the number of pulleys in the different blocks.

21. A man of 12 stone supports a mass of $\frac{1}{2}$ ton by means of two pulley-blocks containing in all nine pulleys. The string which he pulls passes round all of the pulleys, and the lower pulley-block weighs 5 pounds. Taking the free parts of the string as all vertical, and neglecting the weight of the string, find the pressure of the man on the ground.

22. If the man in the preceding question sits on the mass, with what force must he pull at the "Power"-string in order to support himself and the mass?

23. Draw any system of pulleys with vertical strings by means of which a "Power" may balance a "Weight" *eight* times as great, the pulleys themselves being of no appreciable weight.

Through what distance must the point of application of the "Power" move in order to raise the "Weight" through 5 feet?

24. Work the preceding question substituting *seven* for *eight*.

25. In a system of three pulleys, each weighing 1 pound, the top pulley is fixed, the others are movable, and all the strings are attached to a bar which supports the "Weight," the bar itself weighing 6 pounds. If all straight portions of string are vertical, find what "Power" will support a "Weight" of 250 pounds.

Through what distance does the point of application of the "Power" move in raising the "Weight" through 9 inches?

26. In a system of three pulleys the top pulley is fixed, the others are movable, and all the strings are attached to a bar which supports the "Weight," all straight portions of string being vertical. Find how far the "Weight" can be practically raised if the bar is initially 24 feet below the lower movable pulley.

27. In a system of four pulleys, each of weight w and radius a, the top pulley is fixed, the others are movable, and all the strings are attached to a bar which supports the "Weight," the bar itself being of no appreciable weight. If all straight portions of string are vertical, find the relation between the "Power" P and the "Weight" W, and determine the distance between the lines of action of the "Power" and the "Weight" in terms of W, w and a.

28. Find the relation between the "Power" P and the "Weight" W in the third system of pulleys, when there are n pulleys of weight $w_1, w_2, \ldots w_n$, commencing with the lowest, the top pulley being fixed and all straight portions of string being vertical.

29. In the preceding question, find the relation between P and W, when each of the pulleys is of weight w.

30. In a system of n pulleys, each of radius a and of inappreciable weight, the top pulley is fixed, the others are movable, and all the strings are attached to a bar which supports the "Weight," the bar itself being of no appreciable weight. If all straight portions of string are vertical, find the distance between the lines of action of the "Power" and the "Weight."

31. Find the relation between the "Power" P and the "Weight" W in the system of pulleys (known as *The Spanish Barton*) indicated in Fig. 167, the pulleys A_1 and A_2 being of weight w_1 and w_2 respectively.

Through what distance must the "Power" be exerted in raising the "Weight" through a distance of 5 inches?

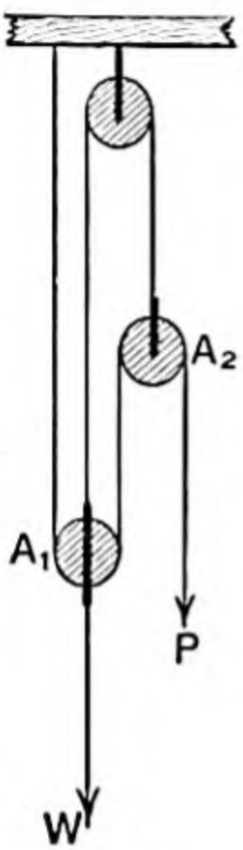


FIG. 167.

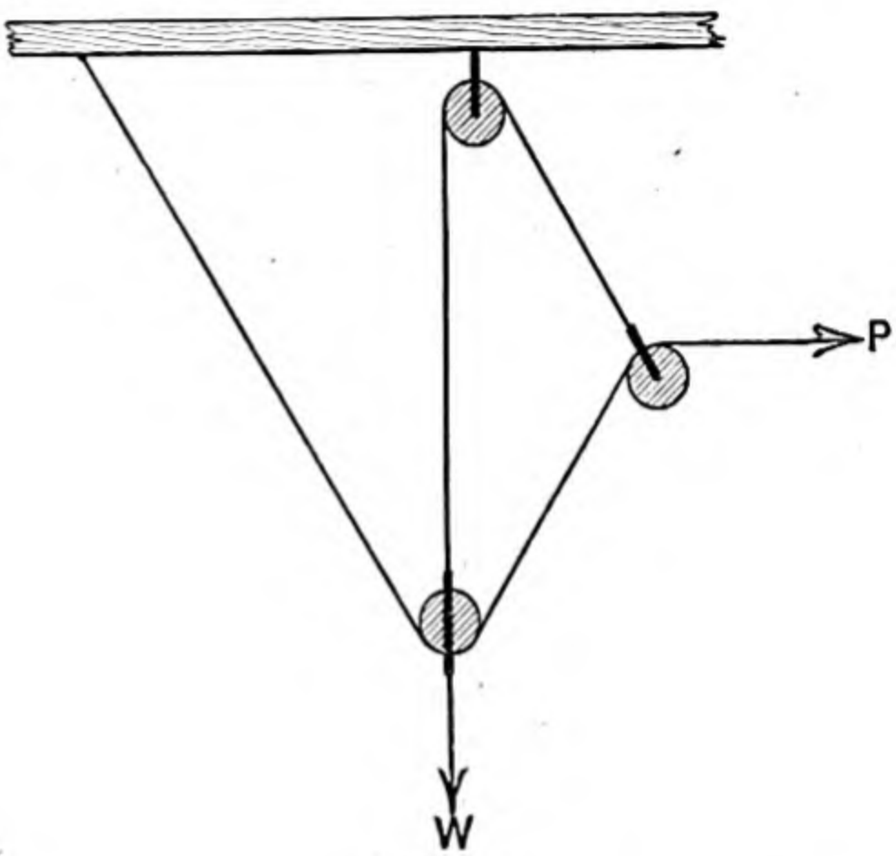


FIG. 168.

32. Find the relation between the "Power" P and the "Weight" W for equilibrium in the system of pulleys indicated in Fig. 168, each pulley being of no appreciable weight, the centre string being vertical, the "Power"-string horizontal, and the angle between the two parts of the string which passes round the lowest pulley being 60°.

33. There are three movable pulleys of no appreciable weight, each hanging by a separate string attached to a fixed supporting beam, and all the strings being vertical except that passing round the highest pulley, the parts of which include an angle of 120°. Show the system in a diagram, and determine the relation between the "Power" P and the "Weight" W.

34. A mass of weight P, hanging freely over a smooth fixed pulley, supports a mass of weight W on a smooth plane inclined to the horizontal at an angle α. Determine the position of equilibrium, and the pressure on the plane.

35. Upon a smooth inclined plane there rests a uniform cube with four of its edges horizontal and one face in contact with the plane; if the force applied to support it acts in a line of greatest slope in that face, find the least inclination of the plane to the horizontal for which the cube will turn about its lowest edge.

36. A uniform square lamina $ABCD$ rests in a vertical plane with one edge AB on a smooth plane inclined at an angle $\alpha(>\frac{\pi}{4})$ to the

horizontal, the corner B being uppermost. The lamina is supported by a force applied at E in BC in a direction parallel to AB. Show that, if each side of the square is of length a, then BE is intermediate in value to $\frac{1}{2}a(1-\cot\alpha)$ and $\frac{1}{2}a(1+\cot\alpha)$.

37. The step—or distance between two consecutive threads—of a smooth screw is 0·187 of an inch, the length of the arm (reckoned from the centre of the screw) on which the effort acts is 25 inches, and the effort is 11·9 pounds' weight; find the resistance, supposing that the screw is on the point of moving.

[Take $\pi = \frac{22}{7}$]

38. If a "Power" of 1 cwt, acting horizontally and at right angles to the extremity of an arm 8 feet 4 inches long, will raise 5 tons by means of a smooth screw, whose axis is vertical and diameter 2 inches, find the inclination to the horizontal of the thread of the screw.

39. The coupling between two railway-carriages consists of a right-handed and left-handed screw at opposite ends of a long bolt, working in nuts attached to the two carriages, each screw having five threads to the inch. Find what force, applied to the end of an arm 18 inches long, will cause the carriages to be drawn together with a force of 1 ton weight, taking the ratio of the circumference to the diameter of a circle as $3\frac{1}{7} : 1$ and supposing the screws to be smooth.

186. Balances.—The various kinds of *weighing-machines* which are constructed for the purpose of determining the weights of substances are of two classes,—those in which the weight of a substance is determined by examining its effect in deflecting a spring, and those in which the weight is determined by applying it to a lever or combination of levers. We shall confine our attention to the more simple examples of those weighing-machines which are constructed on the principle of the lever, and which are usually called *balances;* weighing-machines of the former class are called *spring-balances.*

187. The Common or Roman Steelyard.—This balance consists of a lever shaped somewhat as in Fig. 169 (see next page), possessing a long arm OD, and capable of turning freely in a vertical plane about a fixed fulcrum F, situated towards the thicker end of the lever.

The substance to be weighed is suspended from a point A at the thicker end of the lever, and its weight is determined by observing at what point of the arm OD, a movable mass of weight P must be placed, to allow the lever to rest in equilibrium in a horizontal position.

188. *To graduate the Steelyard.*—When the lever is horizontal, let the horizontal distance between A and F be a, and let x be

the horizontal distance between F and the centre of gravity of the movable mass, so that there may be equilibrium when a mass of weight W is suspended from A.

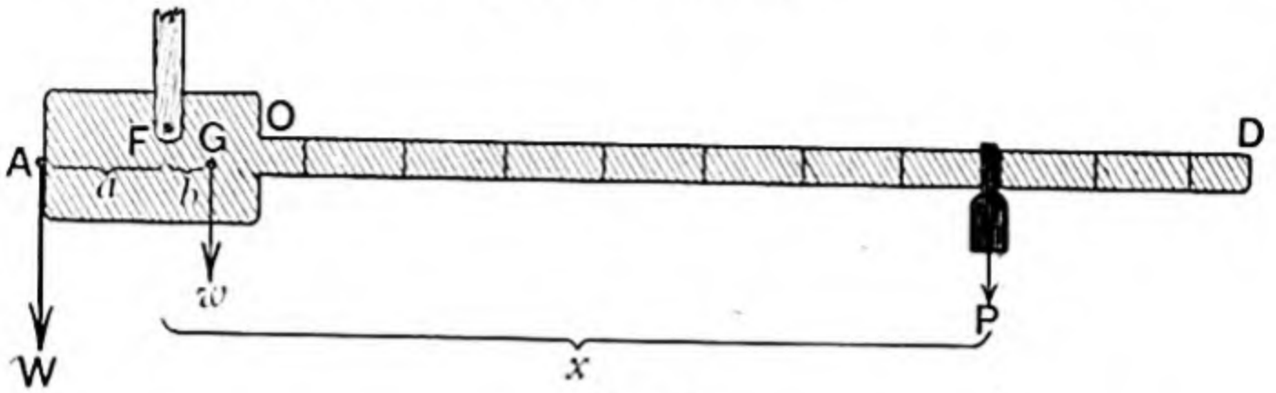


FIG. 169.

Let G be the centre of gravity of the instrument (not including the movable mass) and w its weight, and let the horizontal distance between F and G be b, estimated so that b is negative when G is situated in the shorter arm.

Then, taking moments about F, we have

$$Wa = Px + wb.$$

In this formula the quantities a, P, w, b are constants, and it gives x when W is known, or W when x is known.

Now let x become x' when W is changed into $W + 1$. Then

$$Wa + a = Px' + wb,$$

i.e.

$$Px + wb + a = Px' + wb,$$

$$\therefore x' - x = \frac{a}{P}.$$

Thus the distance between successive graduations is always of constant length $\frac{a}{P}$, and the different values of x, corresponding to successive values of W, form an arithmetic progression of common difference $\frac{a}{P}$.

Example.—Suppose that the lever is a yard long and that the fulcrum is 4 inches from the end A; also that the movable mass is 2 pounds, and that it must be placed at a distance of 6 inches from the end D when weighing 12 pounds suspended from A.

Measuring all weights in pounds' weight and all lengths in inches, and taking moments about the fulcrum, we have

$$4W = 2x + wb.$$

In this equation x is 26 when W is 12

$$\therefore 48 = 52 + wb,$$

$$\therefore wb = -4.$$

The negative result shows that G lies between A and F.
Substituting for wb, the equation connecting W and x becomes

$$4W = 2x - 4,$$

i.e.

$$2W = x - 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(i)}.$$

This gives W when x is known or x when W is known.
Let x become x' when W is changed into $W + 1$.
Then

$$2W + 2 = x' - 2,$$

i.e.

$$(x - 2) + 2 = x' - 2$$

$$\therefore x' - x = 2.$$

Thus the distance between successive graduations, for weighing in pounds, is 2 inches.

In equation (i), the greatest possible value of x is 32, so that the greatest possible value of W is $\frac{32-2}{2} = 15$; also, when W is zero x is 2. Thus the steelyard will weigh anything from 0 to 15 pounds, provided that the movable mass can slide along the 30 inches of the lever which terminate at D.

189. The Danish Steelyard.—This balance consists of a lever shaped somewhat as in Fig. 170, possessing a long arm

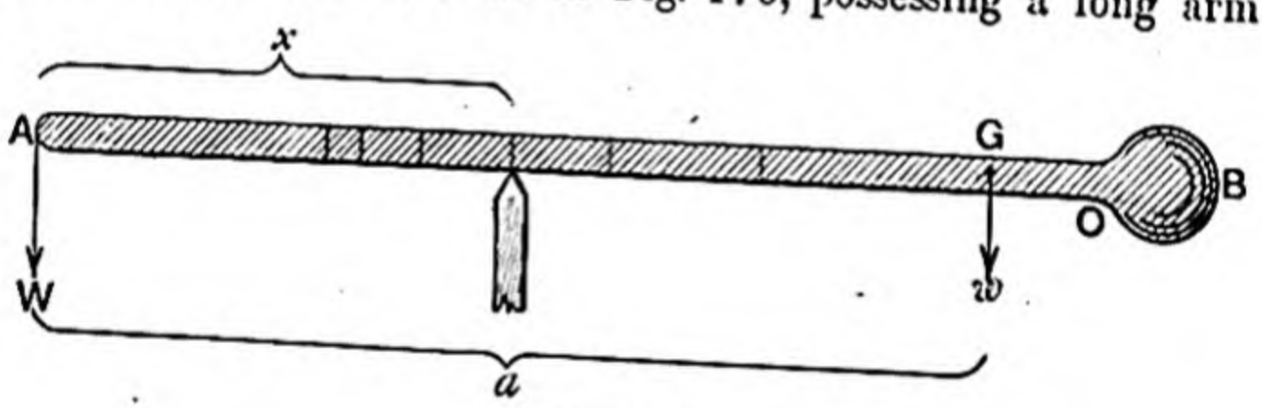


FIG. 170.

AO terminated in a heavy knob B. From the other end A is suspended the substance to be weighed, and its weight is determined by observing about what point of the arm OA the lever will balance.

190. *To Graduate the Steelyard.*—Let G be the centre of gravity of the instrument and w its weight; and, when the lever is horizontal, let a be the horizontal distance between A and G. Let x be the horizontal distance between A and the fulcrum when a mass of weight W is suspended from A.

Then, taking moments about the fulcrum, we have

$$w(a - x) = Wx,$$

$$\therefore (W + w)x = wa.$$ a rect^{e} hyperbola. Cf. log^{c} cur

In this formula the quantities w and a are constants, and it gives x when W is known, or W when x is known. the sli

If x becomes x' when W is changed into $W+1$, we have

$$(W+1+w)x' = wa.$$

$$\therefore \frac{wa}{x'} = W + w + 1$$

$$= \frac{wa}{x} + 1$$

$$\therefore \frac{1}{x'} - \frac{1}{x} = \frac{1}{wa}.$$

Thus the different values of x corresponding to successive values of W form a harmonic progression.

Example.—Suppose that the lever balances about a point 24 inches from A when not in use, and about a point 6 inches from A when a mass of 9 pounds is suspended from A.

Measuring all weights in pounds' weight and all lengths in inches, and noting that $a=24$, we have

$$(W+w)x = 24w.$$

In this equation, x is 6 when W is 9,

$$\therefore (9+w)6 = 24w,$$

$$\therefore w = 3.$$

Thus the equation connecting W and x becomes

$$(W+3)x = 72.$$

This gives x when W is known or W when x is known.

For instance,

when	$W=0$,	$x=24$
,,	$W=1$,	$x=18$
,,	$W=2$,	$x=14\frac{2}{5}$
,,	$W=3$,	$x=12$
,,	$W=4$,	$x=10\frac{2}{7}$
,,	$W=5$,	$x=9$
,,	$W=6$,	$x=8$
,,	etc.,	

or thus,

when	$x=3$,	$W=21$
,,	$x=4$,	$W=15$,
	etc.	

191. The Common Balance.—The common balance consists essentially of a rigid beam AFA', capable of turning freely about a fixed fulcrum F. From one end A is freely suspended a scalepan containing the substance to be weighed, and from the other end A' is freely suspended another scalepan in which is placed a substance of known weight. If the balance is properly constructed, the beam rests with AA' horizontal when, and only when, the weights of the substances in the two scale-

pans are equal. The beam usually carries a long pointer at right angles to AA', for the purpose of indicating when AA' is horizontal or otherwise.

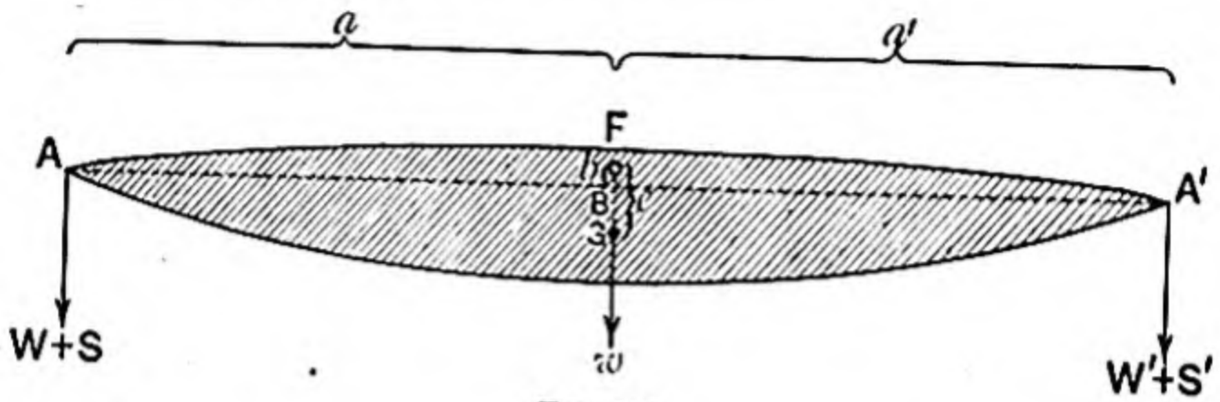


FIG. 171.

The beam is so constructed that it rests with AA' horizontal when there is nothing suspended from A or from A'. This means that the centre of gravity G of the beam lies in the line drawn through F perpendicular to AA'.

Let w be the weight of the beam, S and S' the weights of the scalepans suspended from A and A' respectively, W and W' the weights of the substances placed in those scalepans respectively.

Let FB be drawn perpendicular from F upon AA'. Then AB and $A'B$ are called the arms of the balance. Let FB, FG, AB, $A'B$ be of lengths b, c, a, a' respectively.

192. *The Balance must be* **True.**—Whenever the balance rests with AA' horizontal, the weights W and W' must be equal. Under these circumstances the balance is said to be *true*.

For this to be the case, we must have, taking moments about F,

$$(W+S)a=(W+S')a', \text{ for all values of } W;$$

i.e.

$$W(a-a')=S'a'-Sa, \text{ for all values of } W.$$

This is only possible when $a=a'$ and $Sa=S'a'$. This requires that $a=a'$ and $S=S'$.

Hence, for the balance to be *true*, the arms must be equal and the scalepans of equal weight.

In the two following articles we shall assume that the balance is *true*.

193. *The Balance must be* **Stable.**—Whenever W and W' are equal, the balance must not rest in any position other than that in which AA' is horizontal; if it is placed with AA' not horizontal, then it must quickly return to its position of equilibrium. Under these circumstances the balance is said to be *stable*.

The determination of the time taken for the balance to return to its position of equilibrium is a dynamical question. We may however *assume* that the stability is great or small according as the moment about the fulcrum is great or small.

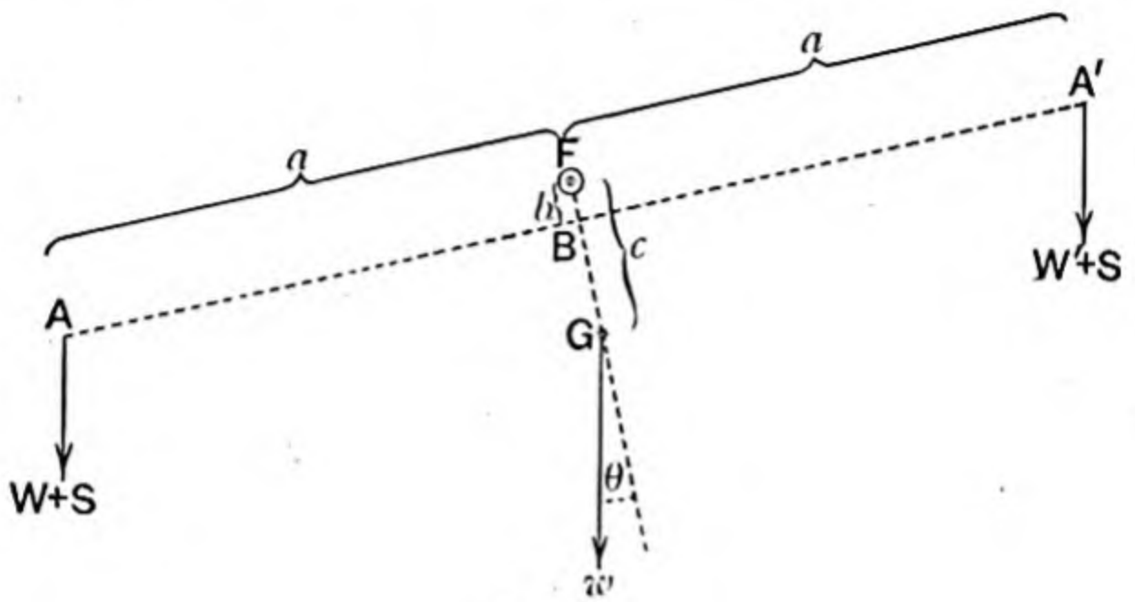


FIG. 172.

Let the balance be placed with AA' inclined at an angle θ to the horizontal, and therefore FBG inclined at an angle θ to the vertical, A' being uppermost and substances of weight W and W' being placed in the scalepans.

Then, the perpendicular from F upon the vertical through A is of length $a\cos\theta - b\sin\theta$.

The perpendicular from F upon the vertical through A' is of length $a\cos\theta + b\sin\theta$.

And the perpendicular from F upon the vertical through G is of length $c\sin\theta$.

Thus the sum of the moments about F of the forces acting upon the beam, estimated in a direction tending to restore AA' to the horizontal position, is

$$(W'+S)(a\cos\theta + b\sin\theta) + wc\sin\theta - (W+S)(a\cos\theta - b\sin\theta)$$
$$= (W'-W)\,a\cos\theta + (W'+W+2S)b\sin\theta + wc\sin\theta.$$

If $W' = W$, the sum of the moments about F becomes

$$\sin\theta\{2(W+S)b + wc\}.$$

Thus, we see that the balance is useless when b and c both vanish, and that the stability increases as the value of $\{2(W+S)b + wc\}$ increases.

194. *The Balance must be* **Sensitive.**—For any slight difference between W and W', the beam must rest in equilibrium

with AA' inclined at an appreciable angle to the horizontal. Under these circumstances the balance is said to be *sensitive*.

Let the balance be in equilibrium when AA' is inclined at an angle θ to the horizontal. Then the sum of the moments about F of the forces acting upon the beam must vanish. Thus the expression

$$(W' - W)a\cos\theta + (W' + W + 2S)b\sin\theta + wc\sin\theta,$$

obtained in the preceding article, must vanish.

$$\therefore \{(W' + W + 2S)b + wc\}\sin\theta = (W - W')a\cos\theta;$$

$$\therefore \frac{a}{(W' + W + 2S)b + wc} = \frac{\tan\theta}{W - W'}.$$

Now $\tan\theta$ must be considerable for a slight difference between W and W'. Hence the value of

$$\frac{a}{2(W + S)b + wc}$$

must be considerable.

Thus, we see that the sensitiveness of the balance increases as a increases and diminishes as $\{2(W + S)b + wc\}$ increases.

195. It appears, then, that the conditions of stability and sensitiveness are to some extent antagonistic. Sufficient sensitiveness and stability can for ordinary purposes be secured by constructing the beam of light material with fairly long arms and making c not very small. It is desirable that b should be zero, for otherwise both the sensitiveness and the stability vary with the load.

The relative importance of sensitiveness and stability depends upon the purpose for which the balance is required. For ordinary commercial purposes extreme sensitiveness may be sacrificed for the sake of rapidity of action, while for chemical research quickness of weighing is of but slight importance and extreme sensitiveness desirable.

***196. False Balance.**—A balance may be false from any one of the following causes :—

(i) The point G may not lie in FB or FB produced.

(ii) The arms a and a' may not be equal.

(iii) The weights of the scalepans S and S' may not be equal.

Let us consider a balance which is false in all three respects. Let x be the distance between the perpendiculars from F and G upon AA', and let the beam be horizontal when substances of weights W and W' are placed in the scalepans.

Then, taking moments about F, we have

$$(W + S)a = wx + (W' + S')a' \quad . \quad . \quad (i).$$

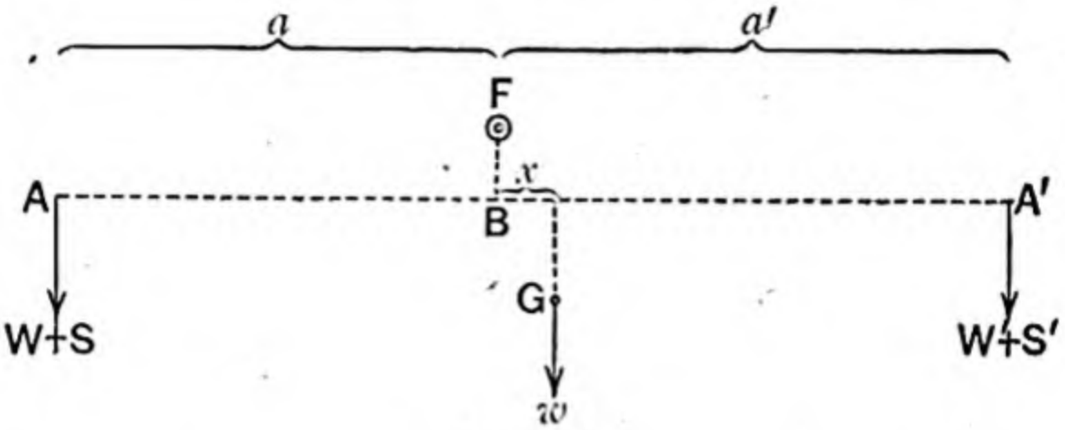


FIG. 173.

1. Suppose that the weights of the scalepans are so adjusted that AA' is horizontal when both pans are unloaded.

Then $$Sa = wx + S'a'.$$

Subtracting this from (i) we have

$$Wa = W'a'.$$

Thus $W = W'$ if $a = a'$. Hence, if the arms are equal, the balance is still a true balance.

If a and a' are not equal, let the substance of weight W be transferred to the other scalepan, and let W'' be the weight of the substance which must be placed in the scalepan left empty, in order that AA' may rest in a horizontal position.

Then $$W''a = Wa'.$$

Hence $$\frac{W}{W''} = \frac{W'}{W}$$

$$\therefore W = \sqrt{W'W''}.$$

Thus, *if the beam is horizontal when both scalepans are unloaded, and if a body is weighed successively in each scalepan, then its real weight is the geometric mean between its apparent weights.*

Also, we have

$$\frac{a}{a'} = \frac{W'}{W} = \frac{W'}{\sqrt{W'W''}} = \sqrt{\frac{W'}{W''}},$$

$$\therefore W' : W'' = a^2 : a'^2.$$

Thus, *if the beam is horizontal when both scalepans are unloaded, and if a body is weighed successively in each scalepan, then the two apparent weights of every body are in a constant ratio, equal to the duplicate ratio of the arms of the balance.*

2. Suppose that the condition 1 is not satisfied, but that the arms are of equal length. Then the equation (i) becomes

$$(W + S)a = wx + (W' + S')a.$$

Also, if, as before, the substance of weight W is weighed in the other scalepan, we have

$$(W'' + S)a = wx + (W + S')a.$$

Hence, subtracting

$$(W - W'')a = (W' - W)a$$
$$\therefore \quad W = \tfrac{1}{2}(W' + W'').$$

Thus, *if the arms of a false balance are equal, and if a body is weighed successively in each scalepan, then its real weight is the arithmetic mean between its apparent weights.*

* **197. Ex. 37.**—*The beam of a tradesman's balance is horizontal when both scalepans are unloaded, but the arms are of unequal lengths a and a'. Show that, if the tradesman weighs out quantities of the same substance which appear to be equal, using alternately each of the scale-pans, then he loses on his transactions, and his actual receipts ought to amount to* $\dfrac{50(a - a')^2}{aa'}$ *per cent more.*

Using the notation of Case 1 of Art. 196, if the tradesman weighs out two quantities, each of which appears to be of weight W, using first one scalepan and then the other, he really gives his customers a quantity of weight $W' + W''$, while he charges for a quantity of weight $2W$.

Now $W' + W'' = W\left(\dfrac{a}{a'} + \dfrac{a'}{a}\right) = W\dfrac{a^2 + a'^2}{aa'}$,

so that $W' + W'' - 2W = W\dfrac{(a - a')^2}{aa'}$,

which is a positive quantity.

Hence he loses on his transactions.

Also,

$$\text{money actually received : money due}$$
$$= 2W : W' + W''$$
$$= 2 : \frac{a^2 + a'^2}{aa'}$$
$$= 2aa' : a^2 + a'^2.$$

Hence his receipts ought to be increased by $\frac{a^2 + a'^2 - 2aa'}{2aa'}$ of their actual amount,

i.e. by $\frac{50(a - a')^2}{aa'}$ per cent.

198. Double Weighing.—The weight of a body can be accurately determined in the following manner: Place the body whose weight is to be determined in one scalepan, and in the other pan put sand or any other suitable material until the beam rests in equilibrium in some observed position. Then, removing the body, place masses of known weight in the first scalepan until the beam rests in its former position. It is clear that, whether the balance is true or not, the weight of the body is equal to the sum of the known weights of the masses which replace it.

When great accuracy is required, this method is used even when the balance is known to be a good instrument.

* **199. Roberval's Balance.**—In the common balance, the scalepans are freely suspended from the extremities of the beam,

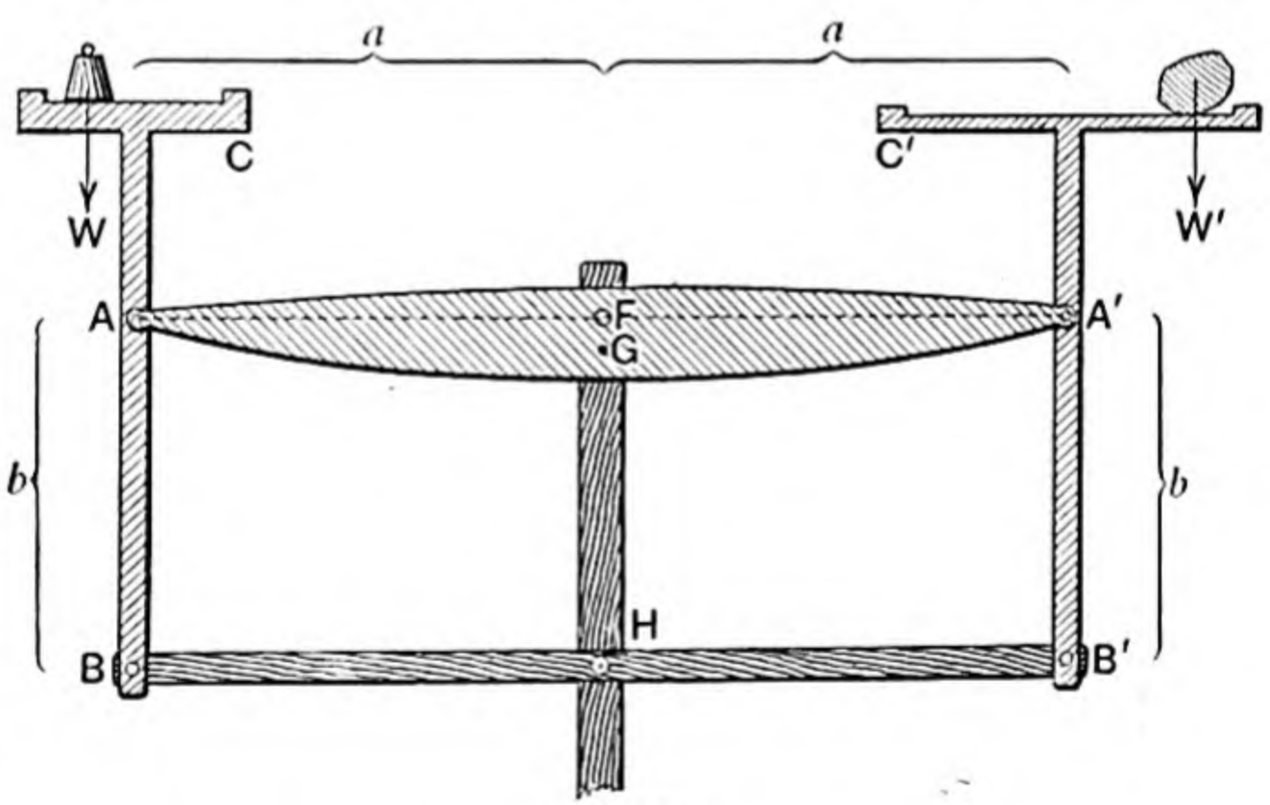


FIG. 174.

and, when the instrument is in use, each scalepan swings about its point of support, until the centre of gravity of the scalepan and all it contains rests vertically below one extremity of the beam. If the scalepan were not free to swing in this manner,

then the arm at which the weight of a substance would act, relatively to the fulcrum, would depend upon its position in the scalepan.

Roberval's Balance is a modification of the common balance, designed so that there may be no swinging of the scalepans, while a substance may have any position in the scalepan.

The beam AA' is capable of turning freely about a fulcrum F situated at the middle point of AA', and its centre of gravity G lies in the straight line drawn through F perpendicular to AA'. A uniform bar BHB', equal in length to AA', is capable of turning freely about its middle point H, which is fixed vertically below F. The piece BAC, holding the scalepan C, is freely hinged at A and B; and the piece $B'A'C'$, holding the scalepan C', is freely hinged at A' and B'; the distances AB, FH, $A'B'$ being all equal in length. The pieces BAC and $B'A'C'$ are of equal weight, but are not necessarily similar in shape.

Suppose that the beam is horizontal when substances of weight W and W' are placed *anywhere* in the scalepans C and C' respectively. Then we can show by the principle of virtual work that $W = W'$.

Let the instrument receive a *slight* virtual displacement, by a rotation of the beam through a small angle in a direction causing A to descend a vertical distance x.

As G is situated in the vertical line through F, its slight virtual displacement is in a horizontal direction, so that no virtual work is done by the weight of the beam.

As the arms FA and FA' are equal, the point A *descends* and the point A' *ascends* through the same vertical distance x.

As $AFHB$ is a parallelogram, the line AB remains vertical, so that every point of the piece BAC *descends* through the same vertical distance x. Similarly, every point of the piece $B'A'C'$ *ascends* through the same vertical distance x.

Thus, if w is the weight of each of the pieces BAC and $B'A'C'$, the method of virtual work gives

$$(W + w)x + (W' + w)(-x) = 0,$$
$$\therefore W = W'.$$

EXAMPLES XXIII

1. Show that, in a common steelyard, the distances between the graduations are proportional to the differences of the weights to which they belong.

2. In Art. 187, if x_0 is the distance of the zero graduation from F, show that the equation $Wa = Px + wb$ may be written in the form $Wa = P(x - x_0)$.

3. A common steelyard is 3 feet long and weighs 4 pounds, its centre of gravity being distant 5 inches from one end. The fulcrum is situated 4 inches from that end, and the mass whose weight is required is suspended from the same end. If the sliding-mass is 3 pounds, find (i) the greatest mass that can be weighed, (ii) the position of the sliding-mass when the instrument is used to weigh 16 pounds, and (iii) the distance between graduations for successive pounds' weight.

4. A common steelyard is 4 feet long and weighs 3 pounds, its centre of gravity being distant 15 inches from the end to which a scalepan of mass one pound is attached. If the movable mass is 3 pounds, find the position of the fulcrum, in order that it may be possible to weigh a mass as great as 20 pounds. Find also the distance between graduations for successive pounds' weight.

5. In a common steelyard, the distance of the fulcrum from the point of suspension of the "Weight" is one inch and the sliding-mass is 2 pounds. The length of the long arm, measured from the fulcrum, is 18 inches; and to weigh 15 pounds, the sliding-mass must be placed 9 inches from the fulcrum. At what distance apart are the graduations marking successive pounds' weight, and what is the weight of the greatest mass that can be weighed?

6. A common steelyard is 3 feet long and weighs 12 pounds, and the fulcrum is distant 4 inches from the end to which the substance to be weighed is attached. If the instrument can weigh no more than 48 pounds, the movable mass being $4\frac{1}{2}$ pounds, find (i) the position of the centre of gravity of the steelyard, (ii) the distance between successive graduations for weighing pounds, and (iii) the position of the movable mass when weighing 16 pounds.

7. A common steelyard, weighing with the scalepan 8 pounds, is 3 feet long, and the fulcrum is situated 2 inches from the point of attachment of the scalepan. The greatest mass that can be weighed is 66 pounds, and the movable mass has to be placed 16 inches from the fulcrum when 30 pounds is put into the scalepan. Find (i) the position of the centre of gravity of the steelyard, (ii) the magnitude of the movable mass, and (iii) the distance between successive graduations for weighing pounds.

8. In a common steelyard, the substance to be weighed is supported at a distance a from the fulcrum, and the weight of the movable mass is of magnitude P. If the movable mass is placed at a distance x from the fulcrum in weighing a mass of weight W, at what distance from the fulcrum must it be placed in weighing a mass of weight W'?

9. A tradesman, using a common steelyard, breaks off a portion from the movable mass. Show that he defrauds his customers.

10. A uniform bar AB of weight w, capable of turning freely about a point F, which divides AB in the ratio $a : b$, is used as a steelyard. The substances to be weighed are suspended from A, and a movable mass of weight P can slide along FB. If W_1 and W_0 are the weights of the greatest and least masses that can be weighed, prove that

$$W_1 - W_0 : P = b : a.$$

11. In the preceding question, if W_1 and W_0 are known, and also a and b, determine the values of w and P. Also, show that, in weighing a mass of weight W, the sliding-mass must be placed at a distance $\frac{W - W_0}{W_1 - W_0} b$ from the fulcrum.

12. In the steelyard of Art. 187, the last graduation from O indicates a substance of weight W_1 suspended from A. Show that, if FD is of length c, the steelyard can be used to weigh substances of greater weight than W_1, by suspending from D a mass of weight $\frac{aW_1}{c}$ for every mass of weight W_1 contained in the substance that is being weighed, the remainder of the weight being indicated, as before, by the position of the movable mass.

13. A weighing machine is constructed, on the principle of the common steelyard, so that for each mass of weight W_1 placed in the scalepan, an additional mass of weight Q has to be suspended from the end of the long arm, which is of length c, whilst the weights of smaller masses in the weighing pan are indicated by the position of a mass of weight P which can slide along the arm. Show that the distances between successive graduations must be $\frac{Qc}{PW_1}$.

14. A common steelyard loses $\frac{1}{n}$th of its weight through wear, the position of its centre of gravity remaining unaltered. Show that the graduations may be corrected by adding to each reading an amount $\frac{1}{n}\frac{x_0}{d}$, where x_0 is the distance of the zero graduation from the fulcrum, and d is the distance between successive graduations.

15. In a Danish steelyard, of mass 2 pounds, the distance between the graduations denoting 7 and 10 pounds' weight is one inch. Find the distance between the latter graduation and the zero graduation.

16. In a Danish steelyard, the graduations denoting 3 and 6 pounds in the scalepan are respectively at distances of 12 and 8 inches from the point of suspension of the scalepan. Find (i) the mass of the instrument (including the scalepan), (ii) the distance of the zero graduation from the point of suspension of the scalepan, (iii) the distance from the same point of the graduation denoting 5 pounds, and (iv) the mass in the scalepan when the instrument balances about a point 6 inches from the point of suspension of the scalepan.

17. In a Danish steelyard, the distance between the zero graduation and the end of the arm is divided into n equal parts. Show that, if w is the weight of the instrument, then the rth graduation from zero denotes a weight $\frac{rw}{n-r}$.

If the fulcrum can be placed only at a graduation, show that the weight of the greatest mass that can be weighed is $(n-1)w$.

18. In a Danish steelyard, of mass one pound, the distance between the zero graduation and the end of the arm is divided into 30 equal parts. Determine all those graduations which indicate that the mass weighed is a whole number of pounds.

19. If a Danish steelyard balances about a point half-way between the zero graduation and that which denotes a mass of weight W, then the substance in the scalepan is of weight $\frac{Ww}{W+2w}$, where w is the weight of the instrument.

20. In a Danish steelyard the distance between the graduations denoting 4 and 5 pounds' weight is one inch and equal to the distance between the graduations denoting 14 and 20 pounds' weight. Find (i) the weight of the instrument, and (ii) the value of the graduation which lies midway between those denoting 5 and 14 pounds' weight.

21. A balance consists of a perfectly uniform isosceles triangle suspended from the vertex, the scalepans being hung from the ends of the base. Masses of unequal weight are placed in the pans and balance. They are interchanged, and to effect equilibrium one ounce has to be added to one of them. Show that one scalepan is half-an-ounce heavier than the other.

22. The only fault in a balance being the inequality in weight of the scalepans, what is the real weight of a body which appears to weigh W_1 when placed in one scalepan, and W_2 when placed in the other? also, what is the difference between the weights of the scalepans?

23. One scalepan of a common balance is loaded, so that a body whose true weight is W appears to weigh $W+w$. What will be the apparent weight of the body if it is placed in the other scalepan?

24. When both scalepans of a common balance are empty, the scales balance; but a body appears to weigh 36·3 pounds when placed in one scalepan and 37·5 pounds when placed in the other scalepan. What is the matter with the balance, and what is the true weight of the body?

25. The arms of a balance are in the ratio 15 : 16. When both scalepans are empty the scales balance. What will be the loss to a tradesman who places articles to be weighed in the scalepan which hangs from the end of the shorter arm, if he is asked for 8 pounds of goods priced at 2s. 6d. per pound?

26. In a false balance, a body whose real weight is W, appears to weigh W' when placed in one scalepan and W'' when placed in the other. Show that the arms of the balance are in the ratio

$$W' - W : W - W''.$$

27. The beam of a tradesman's balance is horizontal when both scalepans are unloaded, but one arm is $1+\alpha$ times as long as the other, α being a small fraction. If the tradesman weighs out what appear to be pounds of the same substance, (i) from the end of the longer arm, (ii) from the end of the shorter arm, and (iii) using alternately each of the scalepans, show that his account ought to be

(i) diminished by $\frac{\alpha}{1+\alpha}$ of the amount, (ii) increased by α of the amount,

(iii) increased by $\frac{\alpha^2}{2(1+\alpha)}$ of the amount, respectively.

28. In a false balance, the weights of the scalepans are at first so adjusted that the beam is horizontal when both pans are empty; but

a portion of matter of weight w is then attached to one of the scalepans. If a body, whose real weight is W, appears to weigh W' when placed in one scalepan and W'' when placed in the other, show that the arms of the balance are in the ratio $W' - W : W - W''$, and that

$$w = \frac{W'W'' - W^2}{W - W''}.$$

29. The fulcrum of a true balance is in the line joining the points of suspension of the scalepans. The beam rests at an inclination θ to the horizontal when a mass of weight w is placed in one of the scalepans. Show that if a further mass of weight $w \sec 2\theta$ is placed in the same pan, then the inclination of the beam to the horizontal will be doubled.

30. In Roberval's Balance, Art. 199, the centre of gravity of the piece BAC is in the line BA, and the centre of gravity of the mass in the scalepan C is at a distance h from the line BA; also, the centre of gravity of the piece $B'A'C'$ is in the line $B'A'$, and the centre of gravity of the mass in the scalepan C' is at a distance h' from the line $B'A'$. Prove that

(i) The horizontal stresses at A and B are each of magnitude $\frac{h}{b}W$.

(ii) The horizontal stresses at A' and B' are each of magnitude $\frac{h'}{b}W'$.

(iii) The vertical stress at A = the vertical stress at $A' = Y$ (say).

(iv) The vertical stress at B = the vertical stress at $B' = Z$ (say).

(v) $W + w = Y + Z = W' + w$, and hence $W = W'$.

CHAPTER XI

FRICTION

200. Pressure Between Bodies in Contact.—We have hitherto supposed, in general, that when two bodies are in contact, the pressure between them at any point of contact is perpendicular to both surfaces at that point. Under these circumstances the bodies are said to be *smooth*.

When a heavy body is at rest on a horizontal plane, the resultant resistance of the plane is equal to the weight of the body, and acts vertically upwards along a line through the centre of gravity of the body. This resistance of the plane is a self-adjusting force of sufficient magnitude to prevent the body from penetrating the plane. If a force is applied vertically downwards upon the body, it still remains in equilibrium, the resistance of the plane being increased. If the body and the plane were ideally smooth, then any force, however slight, applied to the body in any direction not vertical, would cause it to slide along the plane, for the plane is incapable of resisting any tendency of the body to slide over it.

We have been considering an ideal case which never actually occurs. Practically, the horizontal plane is always capable of offering a resistance inclined at an angle to the vertical, and it is found, as the result of experiment, that this angle cannot exceed a certain magnitude depending upon the nature of the surfaces in contact. Thus the resistance of the plane is self-adjusting as regards direction as well as magnitude.

201. Friction and Normal Reaction.—Let a body rest in equilibrium against a rough surface at P. Then the resultant resistance R_1 of the surface may be inclined at an angle θ with the normal PN drawn away from the surface, and is of such

magnitude and in such direction as to balance, if possible, all the other forces which act upon the body.

The force R_1 may be resolved into two components, namely, R along PN and F in the direction of a tangent to the surface at P, where

$$\begin{cases} R = R_1 \cos\theta \\ F = R_1 \sin\theta. \end{cases}$$

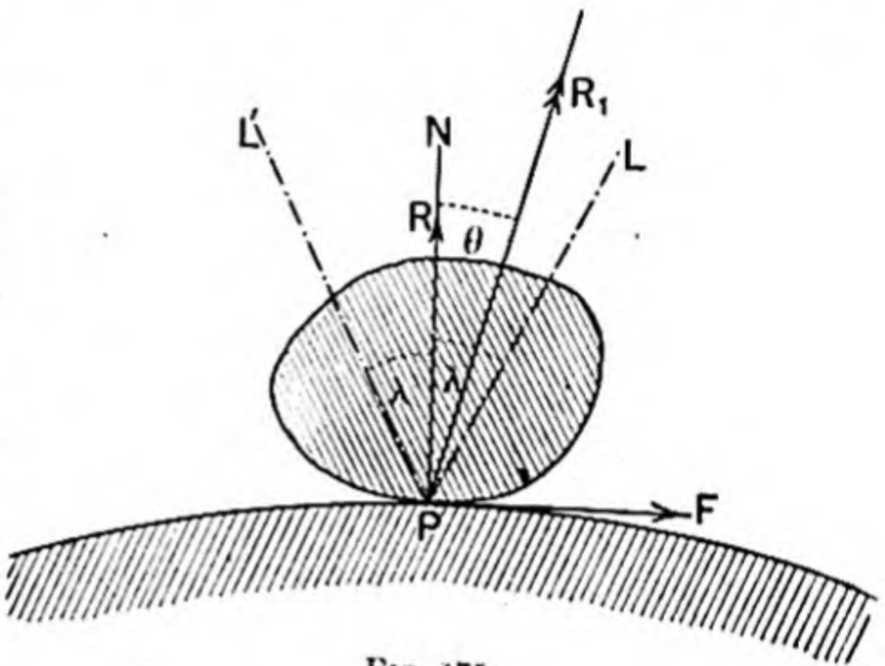


FIG. 175.

The component R is called the *normal reaction* of the surface, and tends to prevent the body from penetrating the surface; the component F is called the *friction*, and tends to prevent the body from slipping over the surface.

The direction of the force F in the tangent plane at P is the direction opposite to that in which the body would begin to slip over the surface, if it were not prevented from so doing by the friction.

202. Angle of Friction.—It is found by experiment that the angle θ can never exceed a certain angle λ depending upon the nature of the surfaces in contact. Thus, if we make $\widehat{NPL} = \lambda = \widehat{NPL'}$, then the resultant resistance must act in a direction intermediate to PL and PL'.

The angle λ is called the *angle of friction*, and is constant for a given pair of surfaces in a given state of polish, being independent of the extent of the area of surface in contact.

203. Suppose that a body rests with a straight edge AB upon a rough plane, the angle of friction between the body and the plane being λ. Then the resultant resistance of the plane acts along a line which intersects AB at some point P between A and B, and which makes an angle not greater than λ with the normal PN drawn away from the plane (Fig. 176).

Draw the normals An and Bn' away from the plane at A and B respectively, and make angles nAl and $n'Bl'$ each equal to λ, so that the angles BAl and ABl' are both obtuse. Then

the resultant resistance of the plane may act anywhere in direction and position between the lines Al and Bl'.

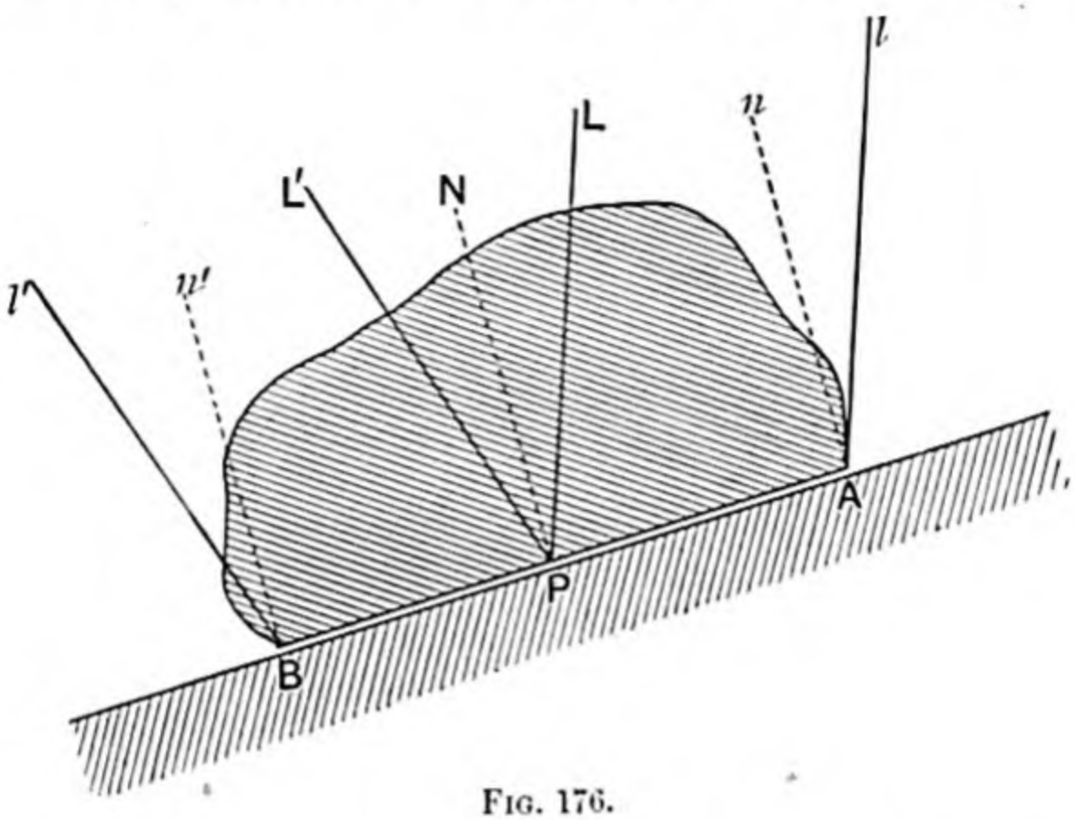


FIG. 176.

204. Coefficient of Friction.—The tangent of the angle of friction is called the *coefficient of friction.*

With the notation of Art. 201 we have $\frac{F}{R} = \tan\theta$, where θ can never be greater than λ. Thus $\frac{F}{R}$ can never be greater than tan λ.

Now tan λ is a constant quantity depending only upon the nature of the surfaces in contact. Denoting its value by μ, we see that $\frac{F}{R}$ can never be greater than the coefficient of friction, μ.

Thus, when two bodies rest in contact, the ratio of the friction between them to the normal pressure is in general less than the coefficient of friction for the two surfaces in contact.

205. Limiting Equilibrium.—When a body is just on the point of slipping over a rough surface, the equilibrium is said to be *limiting.* Under these circumstances, the resultant resistance of the surface at any point of contact makes the extreme angle λ with the normal at that point, and the ratio of the friction to the normal reaction has its maximum value μ.

206. Equilibrium of a Heavy Particle on a Rough Inclined Plane.—*A particle, of given weight W, is placed at O on a rough plane inclined at a given angle α to the horizontal AC, and is sustained by some force P applied to it in some direction OH which is in a vertical plane with AOB, the line of greatest slope of the inclined plane. It is required to examine the conditions of equilibrium.*

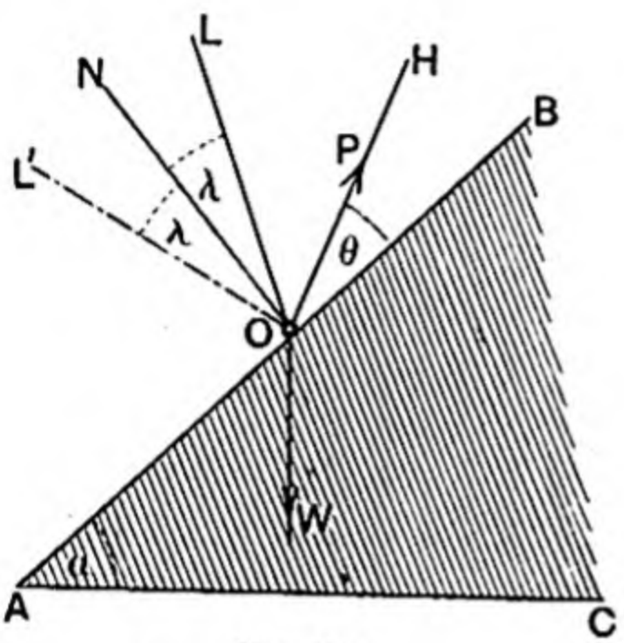


FIG. 177.

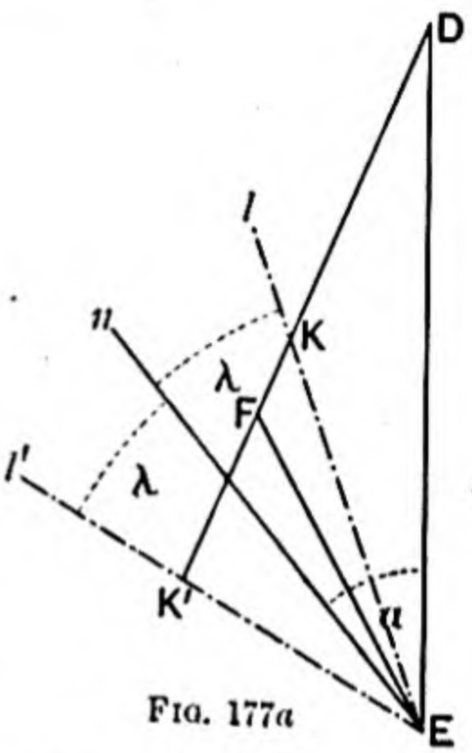


FIG. 177a

Let λ be the angle of friction, and suppose that α is greater than λ.

Draw ON, the normal away from the plane at O, and make $\widehat{LON} = \lambda = \widehat{L'ON}$, so that OL and OL' are within the angles NOB and NOA respectively. Then the force P has to balance a known force W, acting vertically downwards, and a self-adjusting force R_1, which acts in a direction intermediate to OL and OL'.

Draw DE vertically downwards to represent W, and En of unlimited length parallel to ON, so that the angle $DEn = \alpha$. Also draw El and El' parallel to OL and OL' respectively. Then the triangle of forces for the particle will be $DEFD$, where FD is parallel to OH and F lies within the angle lEl'.

Let the straight line drawn through D parallel to HO meet El and El' in K and K' respectively, and let KD and $K'D$ be of lengths Q and Q' respectively.

Then, for equilibrium, the value of P must be between Q and Q'.

The different lines drawn to D from points in El represent

forces which will just prevent the particle from slipping down the plane; and the smallest of these is perpendicular to El, and therefore to OL, and of length $W \sin(\alpha - \lambda)$. Thus the particle is most easily supported on the inclined plane by applying to it a force of magnitude $W \sin(\alpha - \lambda)$ in a direction inclined at an angle λ downwards from OB.

The different lines drawn to D from points in El' represent forces which will be on the point of dragging the particle up the plane; and the smallest of these is perpendicular to El', and therefore to OL', and of length $W \sin(\alpha + \lambda)$. Thus the particle is most easily dragged up the plane by applying to it a force of magnitude $W \sin(\alpha + \lambda)$ in a direction inclined at an angle λ upwards from OB.

The student should consider independently the case in which α is less than λ, so that D lies within the angle lEl'. In this case the particle can rest in equilibrium when there is no force P, the resistance of the plane being W vertically upwards.

If $\alpha = \lambda$, D lies in the line El. In this case, if there is no force P, the particle is on the point of slipping down the plane. Conversely, if the particle is on the point of slipping down the plane when no force P is applied to it, we must have $\alpha = \lambda$. This suggests a practical method for determining the angle of friction by experiment.

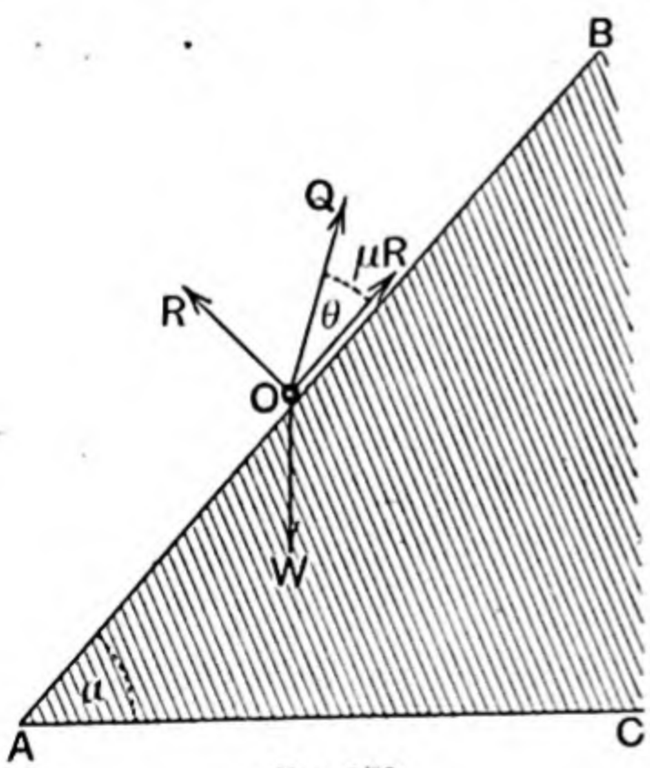


FIG. 178.

Analytically.—Let μ be the coefficient of friction, so that $\tan \lambda = \mu$.

Let the impressed force be applied at an angle θ with the plane, and let Q be its value when the particle is on the point of slipping *down* the plane. Let the normal reaction be of magnitude R; then, as the equilibrium is limiting, the friction is μR *up* the plane.

Resolving along and perpendicular to the plane, we have

$$\begin{cases} \mu R + Q \cos\theta = W \sin\alpha \\ R + Q \sin\theta = W \cos\alpha, \end{cases}$$

$\therefore$ eliminating R,

$$Q(\cos\theta - \mu\sin\theta) = W(\sin\alpha - \mu\cos\alpha),$$

$$\therefore Q = W\frac{\sin\alpha - \mu\cos\alpha}{\cos\theta - \mu\sin\theta}$$

$$= W\frac{\sin\alpha - \tan\lambda\cos\alpha}{\cos\theta - \tan\lambda\sin\theta}$$

$$= W\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)}.$$

The value Q' of the impressed force, when the particle is on the point of sliding *up* the plane, can be deduced from Q by changing the sign of λ.

Thus $$Q' = W\frac{\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}.$$

Hence, for equilibrium, the impressed force must be in value between

$$W\frac{\sin(\alpha - \lambda)}{\cos(\theta + \lambda)} \quad \text{and} \quad W\frac{\sin(\alpha + \lambda)}{\cos(\theta - \lambda)}.$$

For different values of θ, Q is least when $\cos(\theta + \lambda)$ is greatest, *i.e.* when $\theta = -\lambda$, and then $Q = W\sin(\alpha - \lambda)$; also, Q' is least when $\cos(\theta - \lambda)$ is greatest, *i.e.* when $\theta = \lambda$, and then $Q' = W\sin(\alpha + \lambda)$.

*** 207. Equilibrium of a Heavy Body on a Rough Inclined Plane.**—If, instead of a particle, we have a body of finite size resting on a rough inclined plane, the above investigation still holds good for ensuring that the forces acting on the body do not tend to make it slip over the plane; but we have, in addition, to see that they do not tend to overturn the body.

For instance,

A lamina, of weight W, rests in a vertical plane with a straight edge AB along a line of greatest slope of a rough inclined plane. A force P, applied to the lamina at C in direction CH, is gradually increased until motion ensues. It is required to determine whether the lamina slips or turns over.

(i) *Geometrically.*—Let the vertical through G, the centre of gravity of the lamina, meet the line CH in O, and draw ON perpendicular to AB. Let the angle $NOB = \beta$ (Fig. 179).

The resultant of the two forces P and W acts along a line OK through O. Let the angle $NOK = \phi$. Then the effect of increasing P is to increase the angle ϕ.

Now as long as ϕ is less than λ there is no slipping; and as long as ϕ is less than β, there is no turning over round B. Hence the body will slip or turn over round B according as β is $>$ or $<\lambda$.

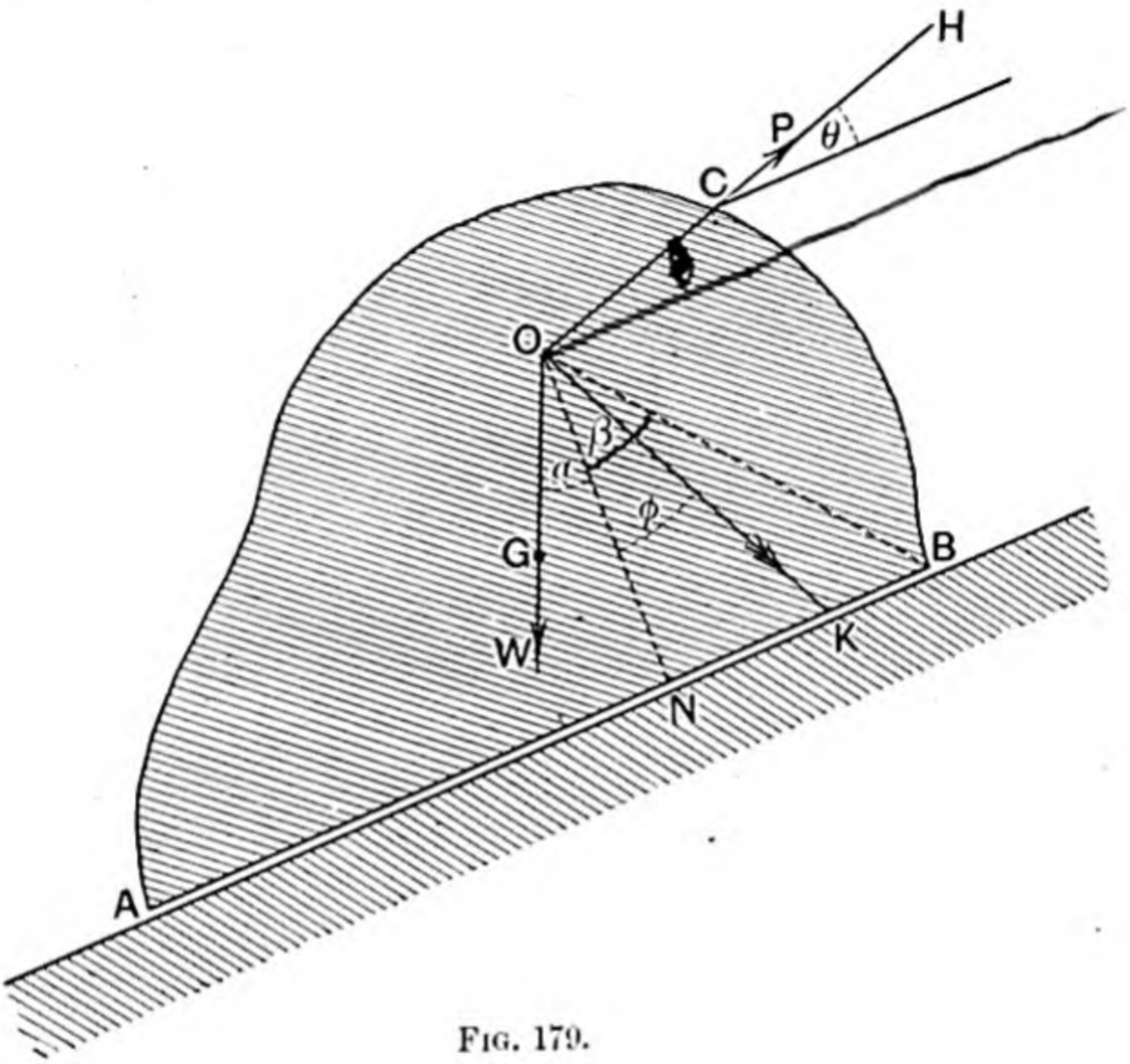


FIG. 179.

(ii) *Analytically.*—Let α be the inclination of the plane to the horizontal, so that ON makes an angle α with the vertical. Let CH be inclined to the plane at an angle θ. Then, if the body is on the point of moving up the plane when the force applied at C is of magnitude P_1, we have by Art. 206

$$P_1 = W\frac{\sin\alpha + \mu\cos\alpha}{\cos\theta + \mu\sin\theta},$$

where μ is the coefficient of friction.

If the body is on the point of turning over round B when the force at C is of magnitude P_2, then the moment of P_2 about B must be equal and opposite to the moment of W about the same point. Now the perpendicular from B upon the line CH is of magnitude $OB\sin\left(\theta+\frac{\pi}{2}-\beta\right) = OB\cos(\theta-\beta)$,

and the perpendicular from B upon the vertical through G is of magnitude $OB \sin(\alpha+\beta)$.

Hence $$P_2 = W\frac{\sin(\alpha+\beta)}{\cos(\theta-\beta)}.$$

If the force at C is gradually increased till motion ensues, the body will slide or turn over according as P_1 is $<$ or $> P_2$;

i.e. according as $\dfrac{\sin(\alpha+\beta)}{\cos(\theta-\beta)}$ is $>$ or $<$ $\dfrac{\sin\alpha+\mu\cos\alpha}{\cos\theta+\mu\sin\theta}$;

i.e. according as $(\cos\theta+\mu\sin\theta)(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$
is $>$ or $<(\sin\alpha+\mu\cos\alpha)(\cos\theta\cos\beta+\sin\theta\sin\beta)$;

i.e. according as $\sin\beta(\cos\alpha\cos\theta-\sin\alpha\sin\theta)$
is $>$ or $<\mu\cos\beta(\cos\alpha\cos\theta-\sin\alpha\sin\theta)$;

i.e. according as $\tan\beta$ is $>$ or $<\mu$.

i.e. according as β is $>$ or $<\lambda$.

208. Ex. 38.—*Two particles P and Q, each of weight W, on two equally rough inclined planes CA and CB of the same height placed back to back, are connected by a light string which passes over the smooth top edge C of the planes. Show that, if the particles are on the point of slipping, then the difference of the inclinations of the planes is double the angle of friction.*

[For a geometrical solution, see the author's *Elementary Geometrical Statics.*]

Let α and β be the inclinations of the planes CA and CB respectively to the horizontal. Let the normal reactions at

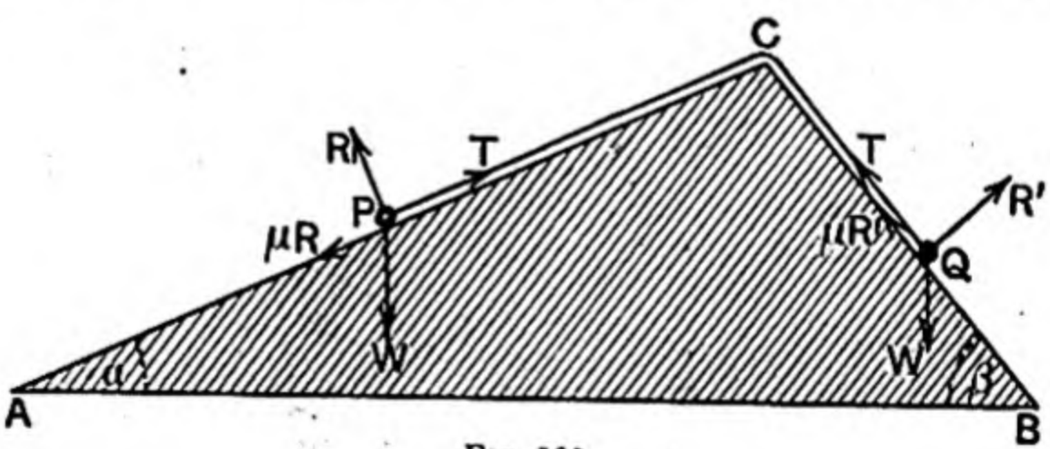


FIG. 180.

P and Q be of magnitude R and R' respectively, and let T be the tension of the string.

Suppose that motion is about to take place in direction ACB. Then the frictions at P and Q are μR and $\mu R'$ respectively in directions CA and BC respectively, where μ is the coefficient of friction.

Resolving along and perpendicular to the plane CA for the equilibrium of the particle P, we have

$$\begin{cases} T = \mu R + W \sin \alpha \\ R = W \cos \alpha. \end{cases}$$

$\therefore$ eliminating R,

$$T = W(\sin \alpha + \mu \cos \alpha)$$

Similarly, for the equilibrium of the particle Q, we obtain

$$T = W(\sin \beta - \mu \cos \beta).$$

Hence, eliminating T,

$$\sin \alpha + \mu \cos \alpha = \sin \beta - \mu \cos \beta ;$$

$$\therefore \sin \beta - \sin \alpha = \mu (\cos \alpha + \cos \beta) ;$$

$$i.e.\ 2 \cos \frac{\beta + \alpha}{2} \sin \frac{\beta - \alpha}{2} = 2\mu \cos \frac{\beta + \alpha}{2} \cos \frac{\beta - \alpha}{2}.$$

As the conditions of the problem do not admit of $\cos \dfrac{\beta + \alpha}{2}$ being zero, we obtain from the equation above,

$$\tan \frac{\beta - \alpha}{2} = \mu = \tan \lambda,$$

where λ is the angle of friction.

$$\therefore \beta - \alpha = 2\lambda.$$

*** 209.** Ex. 39.—*A lamina rests vertically with a straight edge AB in contact with a rough horizontal plane ABD, the coefficient of*

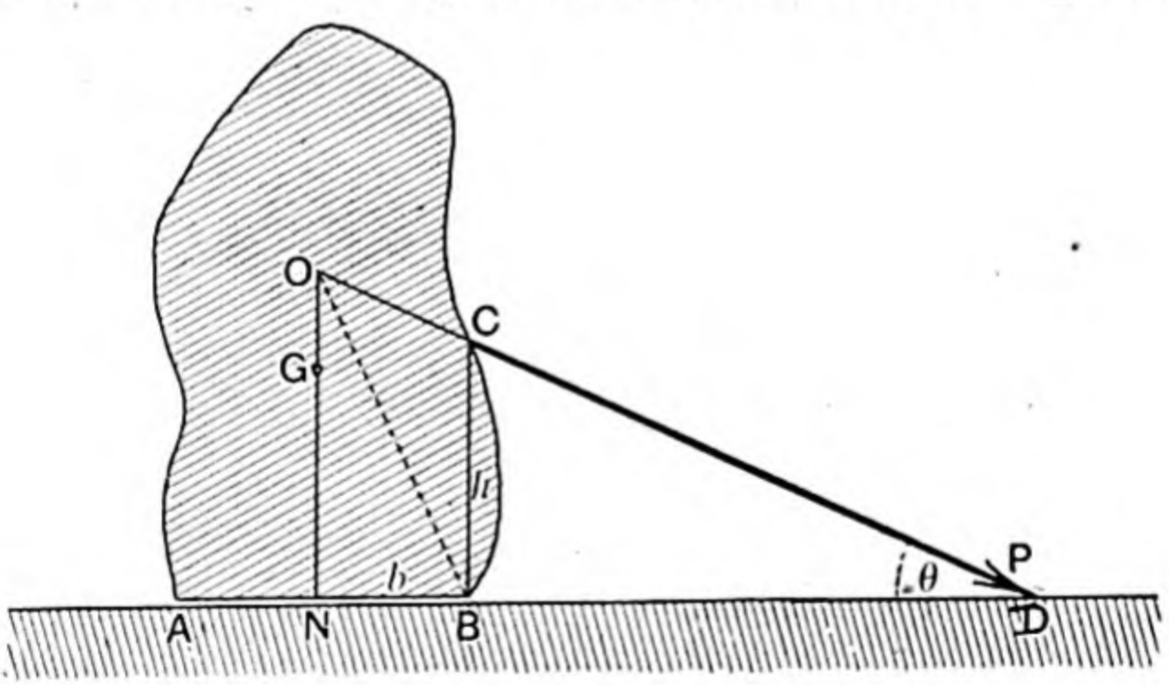


FIG. 181.

friction being μ. One end of a light cord CD is attached to the lamina at a point C, situated vertically above B, and it is desired to tilt the lamina round B by pulling at that extremity D of the

cord which lies on the horizontal plane. The centre of gravity of the lamina is at G, and GN is drawn perpendicular upon AB. If the straight lines BN and BC are of lengths b and h respectively, show that the length of ND must be less than $\frac{b^2}{b-\mu h}$.

(i) *Geometrically.*—Let the lines DC and NG meet at O, and let the angle CDB be of magnitude θ.

Then $ND = b + h\cot\theta,$

and $ON = ND\tan\theta = b\tan\theta + h.$

Now, by Art. 207, in order that the lamina may tilt round B when a force of sufficient magnitude is applied along CD, the angle NOB must be less than the angle of friction,

$$\therefore \tan NOB < \mu,$$
$$\therefore NB < \mu(ON),$$

i.e.
$$b < \mu h + \mu b\tan\theta.$$

If $b < \mu h$, this condition is satisfied for all values of θ. If, however, $b > \mu h$, we must have

$$b - \mu h < \mu b\tan\theta,$$
$$\therefore \cot\theta < \frac{\mu b}{b-\mu h},$$
$$\therefore b + h\cot\theta < b + \frac{\mu b h}{b-\mu h},$$

i.e.
$$ND < \frac{b^2}{b-\mu h}.$$

(ii) *Analytically.*—Let the weight of the lamina be of magnitude W, and suppose that it is on the point of slipping when the force applied is of magnitude P_1. Let R be the normal reaction of the plane, so that the friction is μR, as indicated in Fig. 182.

Resolving vertically and horizontally, we have

$$\begin{cases} P_1\sin\theta + W = R \\ P_1\cos\theta = \mu R, \end{cases}$$

$\therefore$ eliminating R,

$$P_1 = \frac{\mu W}{\cos\theta - \mu\sin\theta}.$$

Suppose that the lamina is on the point of tilting round B when the force applied is of magnitude P_2. Then, taking moments about B, we have

$$P_2 . h \cos\theta = Wb,$$
$$\therefore P_2 = \frac{Wb}{h\cos\theta}.$$

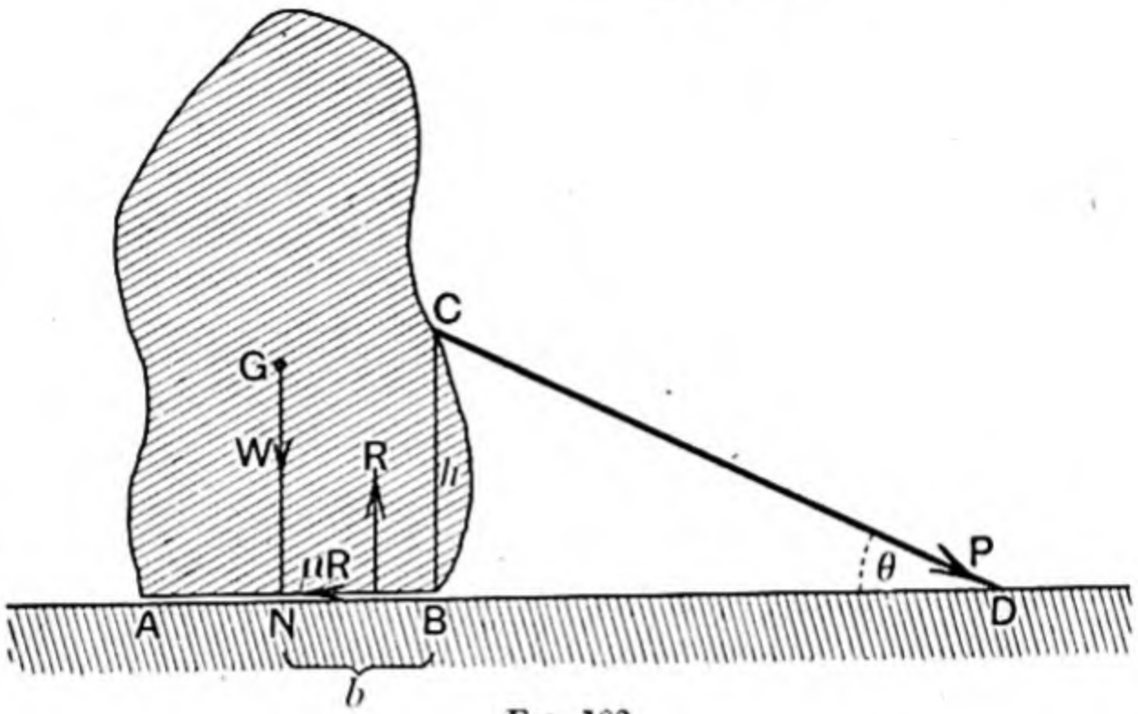


FIG. 182.

As the lamina tilts when the force is sufficiently great, we must have $P_2 < P_1$,

$$\therefore \frac{b}{h\cos\theta} < \frac{\mu}{\cos\theta - \mu\sin\theta}$$
$$\therefore b - \mu b \tan\theta < \mu h,$$
$$\therefore b < \mu h + \mu b \tan\theta.$$

Whence, we obtain the same result as before.

EXAMPLES XXIV

1. A body, of weight W, is placed on a rough *horizontal* plane, the coefficient of friction being $\frac{1}{3}$. Find
   (i) The least horizontal force which will move the body.
   (ii) The least force which, acting in an upward direction at an angle of 30° with the horizontal, will move the body.
   (iii) The magnitude and direction of the total resistance, when a force of magnitude $\frac{1}{4}W$ is applied in an upward direction making an angle of 30° with the horizontal.
   (iv) The magnitude and direction of the least force necessary to move the body.

2. A body, of weight W, is placed on a rough plane inclined to the horizontal at an angle of 30°, the coefficient of friction being $\frac{1}{3}$. Find
   (i) The least horizontal force which will move the body *up* the plane.
   (ii) The least horizontal force which will prevent the body from slipping *down* the plane.
   (iii) The magnitude and direction of the least force necessary to move the body *up* the plane.

(iv) The magnitude and direction of the least force necessary to prevent the body from slipping *down* the plane.

3. A body, of weight W, is placed on a rough plane inclined to the horizontal at an angle of 15°, the coefficient of friction being $\frac{1}{3}$. Find

(i) The least force which will produce motion when acting *up* the plane.

(ii) The least force which will produce motion when acting *down* the plane.

(iii) The magnitude and direction of the least force necessary to move the body *up* the plane.

(iv) The magnitude and direction of the least force necessary to move the body *down* the plane.

4. A body, of weight W, rests on a rough plane inclined at an angle α to the horizontal. If a force Q, acting up and parallel to the plane, is just sufficient to prevent the body from slipping down, find the force, acting in the same direction, which will just cause the body to be on the point of moving up the plane.

5. A body is supported on a rough plane, inclined at an angle α to the horizontal, by means of a force acting up and parallel to the plane. If the greatest and least values of the supporting force consistent with equilibrium are P and Q respectively, find the weight of the body and the coefficient of friction between the body and the plane.

6. A body, of weight W, is supported on a rough inclined plane by means of a force acting up and parallel to the plane. If the greatest and least values of the supporting force consistent with equilibrium are P and Q respectively, find the coefficient of friction and the inclination of the plane to the horizontal.

7. A body, of mass 16 pounds, rests on a rough plane inclined at an angle of 30° to the horizontal. If a force of 2 pounds' weight, acting up and parallel to the plane, is just sufficient to prevent the body from slipping down, find the force, acting in the same direction, which will cause the body to be on the point of moving up the plane.

8. A body, of weight W, is placed on a rough plane of height 5 feet and length 13 feet, the coefficient of friction being $\frac{2}{3}$; find the magnitude of the horizontal force which will just suffice to drag the body *up* the plane.

Find also the magnitude of the horizontal force which will just prevent the body from sliding down.

9. A body, of weight W, is just on the point of slipping when placed on a rough plane inclined to the horizontal at an angle α. Find the coefficient of friction between the body and the plane, and determine the magnitude of the force which, acting parallel to the plane, will just suffice to move the body up the plane.

Determine also the magnitude and direction of the least force necessary to move the body up the plane.

10. A body is just prevented from slipping down a rough plane inclined at an angle α to the horizontal, the angle of friction being λ, by means of a force applied to it in a direction making a given angle with the vertical. Show that the same force applied at the same inclination to the vertical would support the same body on a smooth plane inclined to the horizontal at an angle $\alpha - \lambda$.

11. A body is supported on a rough plane, whose inclination to the *vertical* is equal to twice the angle of friction, by means of a force acting parallel to the plane. If this force is gradually increased, show that the body will begin to move up the plane as soon as the impressed force exceeds the weight of the body.

12. A body is supported on a rough inclined plane by means of a force equal to its own weight. Show that, if the inclination of the plane to the *vertical* is greater than the angle of friction, then the force must be applied in some direction within an angle equal to four times the angle of friction.

13. A body is placed on a rough plane. Prove that no force, however great, applied towards the plane at an angle with the normal less than the angle of friction, can push the body up the plane.

14. A body is supported on a rough plane by a force acting up and parallel to the plane. It is on the point of moving up the plane when the force has the value Q, and on the point of slipping down when the force has the value Q'. Show that a force of magnitude $\frac{1}{2}(Q+Q')$, applied in the same direction, would support the same body on a smooth plane of the same inclination.

15. A heavy body is placed on a rough *horizontal* plane, the angle of friction being λ, and is just on the point of moving when a force P is applied to it in a direction inclined at an angle λ to the upward vertical. Show that the force which is just sufficient to move it, when applied at an angle 3λ with the downward vertical, is also of magnitude P. Prove also that if the body is on the point of moving when under the influence of a horizontal force, then the total resistance of the plane is of magnitude $2P$.

16. A gradually increasing horizontal force is applied to a heavy body placed upon a rough horizontal plane, and when the body begins to move, the total resistance of the plane is of magnitude R. The body is also on the point of moving when a force of magnitude $\frac{1}{2}R$ is applied to it in a direction making an angle θ with the downward vertical. Show that the coefficient of friction is equal either to $\tan\frac{1}{3}\theta$ or to $\tan(\pi-\theta)$, the latter result being inadmissible if θ is acute.

17. A is the lowest point of a rough circular hoop fixed in a vertical plane. A small ring, of weight W, is threaded on the hoop at P, where the arc AP subtends an angle α at the centre of the hoop. Find the magnitude of the least horizontal force which will support the ring, the angle of friction between the ring and the hoop being λ.

18. A small ring, of weight W, is sustained in limiting equilibrium on a rough circular hoop, fixed in a vertical plane, by means of a force $W\sin\alpha$ acting at an angle α with the horizontal and upwards. The angle of friction between the ring and the hoop being λ, find the positions of limiting equilibrium.

19. A small ring P, of inappreciable weight, is capable of sliding on a rough straight piece of wire ACB, which is fixed in a position inclined at an angle α to the vertical. A fine light string has one extremity attached to the ring P, passes through a small smooth ring D, fixed at a distance a vertically below C, and supports at its free end a mass of weight W. If μ is the coefficient of friction between the ring and the wire, show that there is a portion of wire,

of length $2\mu a \sin\alpha$, at any point of which the ring can rest in equilibrium, and that beyond each end of this portion equilibrium is impossible.

20. ABC is a rough wire fixed in a vertical plane, the portions AB and BC being straight and inclined at the same angle α to the horizontal, and B being at a higher level than A and C. P and Q are two small rings which can slide on the wire, P on AB and Q on BC. A fine light string connecting P and Q passes through a small smooth ring R, to which a mass of weight W is attached. If λ is the angle of friction between the rings and the wire, and if P and Q are each on the point of sliding *down*, find the tension of the string in terms of W, α, and λ, the weights of the rings P, Q, and R being inappreciable.

21. A small ring P, of no appreciable weight, can slide on a rough circular hoop of centre O. It is connected with a fixed point A of the hoop by means of a light elastic string in a state of tension. Show that AP cannot rest inclined to OA at an angle greater than the angle of friction.

22. AB is a line of greatest slope of a rough plane inclined at an angle α to the horizontal, B being at a higher level than A. At B the inclined plane adjoins an equally rough horizontal plane BC. Two equal heavy particles, one on AB and the other on BC, are connected by a fine light string which passes over a small smooth pulley at B. Show that, if the particles are on the point of motion, then the angle of friction is $\frac{1}{2}\alpha$.

23. Two equally rough planes, inclined at angles α_1 and α_2 to the horizontal, and having the same altitude, are placed back to back. Two particles, of weights W_1 and W_2, placed one on each plane, are connected by a fine light string which passes over the smooth top edge of the planes. Show that, if the particles are on the point of moving, then the coefficient of friction must be

$$\frac{W_1 \sin\alpha_1 \sim W_2 \sin\alpha_2}{W_1 \cos\alpha_1 + W_2 \cos\alpha_2},$$

and the tension of the string is of magnitude

$$\frac{W_1 W_2 \sin(\alpha_1+\alpha_2)}{W_1 \cos\alpha_1 + W_2 \cos\alpha_2}.$$

24. In the preceding example, if the tension of the string is of magnitude T, show that the cosine of the angle of friction is

$$\frac{W_1 W_2 \sin(\alpha_1+\alpha_2)}{T\sqrt{W_1^2+W_2^2+2W_1W_2\cos(\alpha_1+\alpha_2)}}.$$

25. A force P_1 is just sufficient to drag a heavy body up a rough plane inclined to the horizontal at an angle α_1, the force acting parallel to the plane. A force P_2 is just sufficient to prevent the same body from slipping down an equally rough plane inclined to the horizontal at an angle α_2, the force acting parallel to the plane. Show that the coefficient of friction is

$$\frac{P_1 \sin\alpha_2 - P_2 \sin\alpha_1}{P_1 \cos\alpha_2 + P_2 \cos\alpha_1},$$

and that the body is of weight

$$(P_1 \cos\alpha_2 + P_2 \cos\alpha_1)\operatorname{cosec}(\alpha_1+\alpha_2).$$

26. In the preceding example, if the body is of weight W, show that the cosine of the angle of friction is

$$\frac{W\sin(\alpha_1+\alpha_2)}{\sqrt{P_1^2+P_2^2+2P_1P_2\cos(\alpha_1+\alpha_2)}}.$$

27. A lamina rests in a vertical plane with a straight edge AB along the line of greatest slope of a rough inclined plane, B being at a higher level than A. The centre of gravity of the lamina is at G, and AG makes an angle α with the normal to the plane. If the inclination of the plane to the horizontal is gradually increased till motion takes place, show that the lamina will slide or tilt round A according as α is greater or less than the angle of friction.

28. A uniform rectangular lamina, of length a and breadth b, rests in a vertical plane, with one of its short sides along the line of greatest slope of a rough inclined plane, the coefficient of friction being μ. If the inclination of the plane to the horizontal is gradually increased till motion takes place, show that the lamina will slide or topple over according as b is greater or less than μa.

29. A uniform triangular lamina ABC rests vertically with the side BC along the line of greatest slope of a rough inclined plane, C being at a higher level than B. From E, the middle point of AC, a straight line EK is drawn perpendicular upon BC. If the inclination of the plane to the horizontal is gradually increased till motion takes place, show that the lamina will slide or tilt round B according as the angle BEK is greater or less than the angle of friction.

In particular, if $BA=BC$, or if the angle B is right, show that the lamina will slide or topple over according as the angle A is greater or less than the angle of friction.

30. A lamina, whose centre of gravity is at G, rests vertically with a straight edge AB in contact with a rough horizontal plane, the coefficient of friction being μ. A force applied to the lamina at C, in direction parallel to AB, is gradually increased until motion takes place. Straight lines CM and GN are drawn perpendicular upon AB, and the straight lines BN and CM are of lengths b and h respectively. Show that the lamina will slide or tilt round B according as b is greater or less than μh.

31. A uniform triangular lamina ABC rests vertically with the side BC on a rough horizontal plane. E is the middle point of AC and F divides AB in the ratio $2:1$. From E a straight line EK is drawn perpendicular upon BC. A force applied to the lamina at F, in direction parallel to CB, is gradually increased until motion takes place. Show that the lamina will slide or tilt round B according as the angle BEK is greater or less than the angle of friction.

In particular, if $BA=BC$, or if the angle B is right, show that the lamina will slide or topple over according as the angle A is greater or less than the angle of friction.

32. A rectangular block $ABCD$, whose height is double its base, stands with its base AD on a rough horizontal floor, the coefficient of friction being $\frac{1}{3}$. If it is pulled by a horizontal force at C till motion takes place, determine whether it will slip on the floor or begin to turn over round D.

33. A uniform square lamina $ABCD$ rests vertically with its side AB upon a rough horizontal plane, the coefficient of friction being μ. A force is applied at D in an upward direction, making an angle α with DC. If this force is gradually increased until motion takes place, show that the lamina tilts round B or slides over the plane according as $\mu(2+\tan\alpha)-1$ is positive or negative.

34. A uniform square lamina $ABCD$ rests vertically with its side AB upon a rough horizontal plane, the coefficient of friction being $\frac{1}{3}$. A fine string is attached at the corner D, passes over a small smooth peg P fixed at a point vertically over C, and is pulled at its other extremity by a gradually increasing force till motion ensues. Show that the lamina slides along the plane or tilts round B according as BC is greater or less than PC.

35. In the preceding question, if the lamina is of weight W, find the least force which will produce motion

(i) when $PC=\frac{3}{4}BC$;

(ii) when $PC=\frac{1}{3}BC$.

36. In Ex. 39, Art. 209, a force is applied to the end of the string for the purpose of dragging the lamina over the plane without upsetting it. Show that this is impossible unless $b>\mu h$, and that if this condition is satisfied, the string must be at least of length

$$\frac{h}{b-\mu h}\sqrt{(1+\mu^2)b^2-2\mu bh+\mu^2h^2}.$$

37. A lamina rests vertically with a straight edge AB in contact with a rough horizontal plane ABD, the coefficient of friction being μ. An attempt is made to tilt the lamina round B by applying a force to it at a point C, such that the angle DBC is of magnitude α and the distance BC of length c. The vertical through the centre of gravity of the lamina meets AB at N, so that NB is of length b. Show that (i) if $\mu c>b(\sin\alpha-\mu\cos\alpha)$, then, to overturn the lamina with the least exertion, the force should be applied in a direction perpendicular to BC; while (ii) if $\mu c<b(\sin\alpha-\mu\cos\alpha)$, then, to overturn the lamina with the least exertion, the force should be applied in a direction making an angle

$$\tan^{-1}\frac{b-\mu c\sin\alpha}{\mu(b+c\cos\alpha)}$$

with the horizontal.

38. A lamina, whose centre of gravity is at G, rests vertically with a straight edge AB along the line of greatest slope of a rough inclined plane, so that A is at a higher level than B. A force, applied to the lamina at C, in direction parallel to AB, is gradually increased until motion takes place. Straight lines CH and GK are drawn perpendicular upon AB, and the straight lines CH, GK, and BK are of lengths h, k, and b respectively. Show that, if α is the inclination of the plane to the horizontal and μ the coefficient of friction, then the lamina will slide down the plane or tilt round B according as $h\tan\alpha+b$ is greater or less than $k\tan\alpha+\mu h$.

39. A lamina, whose centre of gravity is at G, rests vertically with a straight edge AB along the line of greatest slope of a rough inclined plane, so that B is at a higher level than A. It is prevented

from slipping down the plane by a light string CD, attached to the lamina at C and to a fixed point D fixed relatively to the plane, so that CD is parallel to AB. Straight lines CH and GK are drawn perpendicular upon AB, and the straight lines CH, GK, BK are of lengths h, k, b respectively. Show that, if h is greater than k, and if μ is the coefficient of friction, then the greatest inclination of the plane to the horizontal consistent with equilibrium is

$$\tan^{-1}\frac{b+\mu h}{h-k}.$$

40. A lamina, whose centre of gravity is at G, rests vertically with a straight edge AB along the line of greatest slope of a rough inclined plane, so that B is at a higher level than A. The lamina is prevented from slipping down the plane by a fine string CD attached to the lamina at C and to a point D fixed in the surface of the plane, the angle DBC being a right angle. GK is drawn perpendicular upon AB, and the straight lines BC, GK, KB, BD are of lengths h, k, b, d respectively. Show that, if μ is the coefficient of friction, then the greatest inclination of the plane to the horizontal consistent with equilibrium is

$$\tan^{-1}\frac{(b+\mu h)d+\mu bh}{(h-k)d-\mu hk}.$$

210. Equilibrium of a Rigid Body resting at one Point against a Rough Surface.—Let a rigid body rest in equilibrium against a rough surface at P, the area in contact being so small that it may be treated as a point; and suppose that, in addition to the resistance of the surface, a given system of forces acts upon the body in a plane containing the point P.

Let λ be the angle of friction. Draw the normal PN away from the surface at P, and make $\widehat{NPL}=\lambda=\widehat{NPL'}$ (Fig. 175). Then the total resistance must act through P in a direction intermediate to PL and PL', and may be of any magnitude.

Hence it is necessary and sufficient for equilibrium that the given system of forces should reduce to a resultant acting through P in a direction intermediate to LP and $L'P$.

Analytically.—The sum of the moments of the given system of forces about P must be zero. Also, if X and Y are the sums of the resolved parts of the given system of forces estimated in direction PN and perpendicular to PN respectively, then X must be negative and the ratio of Y to X must be numerically less than the coefficient of friction μ.

***211.** Ex. 40.—*A heavy beam AB is supported in a given position inclined to the horizontal, with one end B resting on a rough horizontal plane, by means of a force applied at A. It is*

required to determine the limitations upon the direction of the applied force.

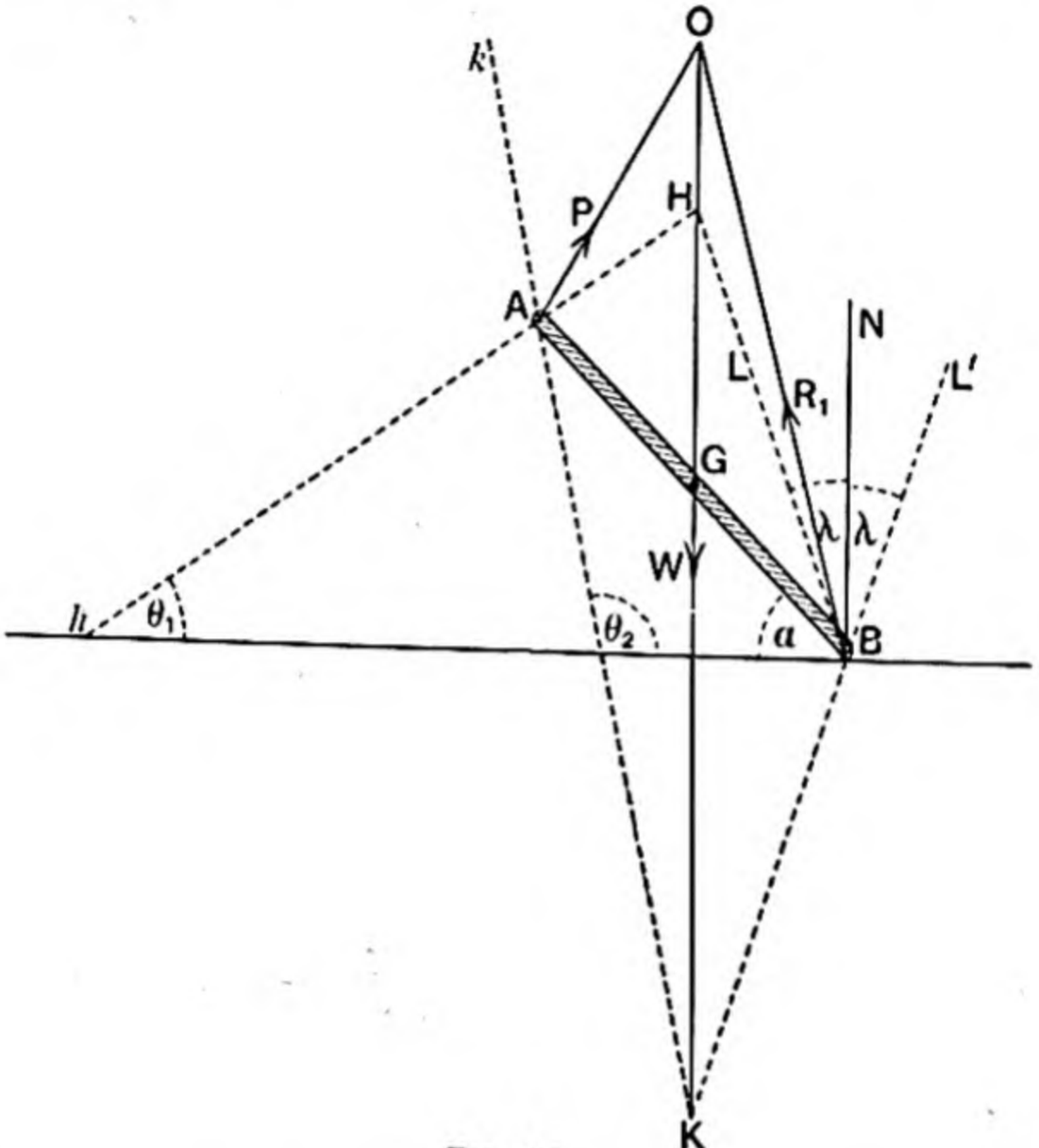


FIG. 183.

(i) *Geometrically.*—Let G be the centre of gravity of the beam and W its weight, and let AG and GB be of lengths a and b respectively. Let the vertical through G, and the line of action of the force P applied at A, meet in O. Then BO is the line of action of the resultant resistance of the horizontal plane.

Draw BN vertically upwards, and make $\widehat{LBN}$ = angle of friction $(\lambda) = \widehat{NBL'}$. Then BO must lie between BL and BL'. Hence, if the vertical through G meets BL and $L'B$ produced in H and K respectively, we see that O must not lie between H and K, but must be above H or below K.

Produce HA and KA to h and k respectively. Then we see that the force applied at A must be in a direction either between AH and Ak or between Ah and AK. Also, as its direction

must be such as to balance the weight of the rod in its tendency to produce rotation about B, we see that it must be in a direction between AH and Ak.

Let AH and Ak be inclined at angles θ_1 and θ_2 respectively to the horizontal, as indicated in the figure. Also let α be the given inclination of the rod to the horizontal. Then

$$\begin{aligned}\frac{a}{b} = \frac{AG}{GB} &= \frac{AG}{GH}\cdot\frac{GH}{GB}\\ &= \frac{\sin AHG}{\sin GAH}\cdot\frac{\sin GBH}{\sin GHB}\\ &= \frac{\cos\theta_1}{\sin(\alpha+\theta_1)}\cdot\frac{\cos(\alpha+\lambda)}{\sin\lambda}\\ &= \frac{\cos\theta_1\cos\alpha\cos\lambda - \cos\theta_1\sin\alpha\sin\lambda}{\sin\alpha\cos\theta_1\sin\lambda + \cos\alpha\sin\theta_1\sin\lambda}\\ &= \frac{1-\mu\tan\alpha}{\mu\tan\alpha+\mu\tan\theta_1},\end{aligned}$$

where μ is the coefficient of friction.

Hence

$$\tan\theta_1 = \frac{b-\mu(a+b)\tan\alpha}{\mu a}.$$

Similarly,

$$\begin{aligned}\frac{a}{b} &= \frac{AG}{GK}\cdot\frac{GK}{GB}\\ &= \frac{\sin AKG}{\sin GAK}\cdot\frac{\sin GBK}{\sin GKB}\\ &= \frac{-\cos\theta_2}{\sin(\theta_2+\alpha)}\cdot\frac{\cos(\lambda-\alpha)}{\sin\lambda}\\ &= -\frac{\cos\theta_2\cos\alpha\cos\lambda + \cos\theta_2\sin\alpha\sin\lambda}{\sin\theta_2\cos\alpha\sin\lambda + \cos\theta_2\sin\alpha\sin\lambda}\\ &= -\frac{1+\mu\tan\alpha}{\mu\tan\theta_2+\mu\tan\alpha}.\end{aligned}$$

Hence

$$\tan\theta_2 = \frac{b+\mu(a+b)\tan\alpha}{-\mu a}.$$

(ii) *Analytically.*—Let the force P be applied at A in a direction making an angle θ with the horizontal, as indicated

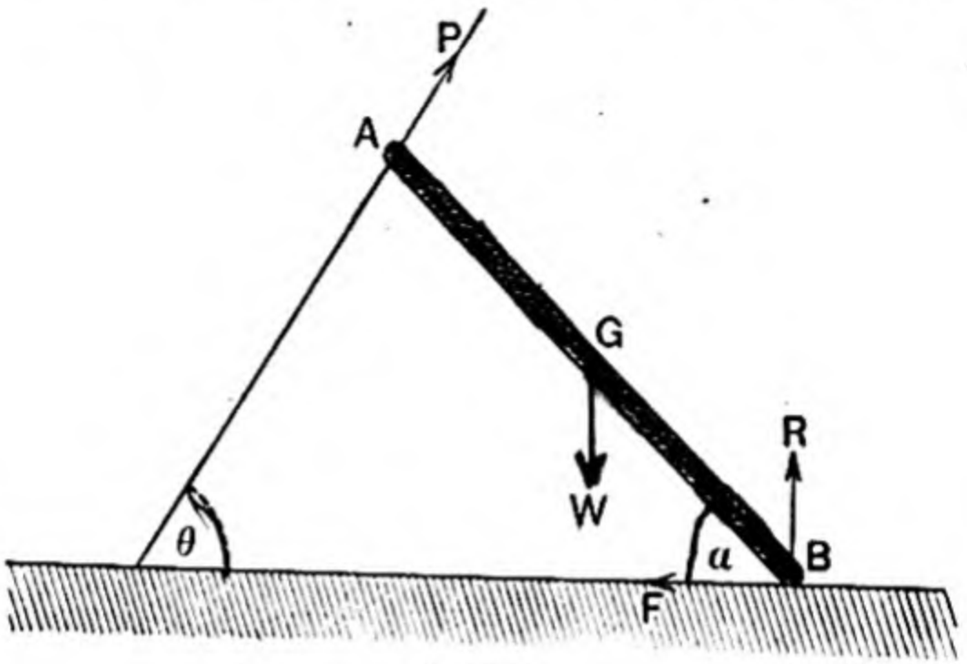


FIG. 184.

in Fig. 184. Let R and F be the normal reaction and the friction respectively at B.

Then, resolving vertically and horizontally, and taking moments about B, we have

$$\begin{cases} P\sin\theta + R = W \\ P\cos\theta = F \\ P(a+b)\sin(a+\theta) = Wb\cos a, \end{cases}$$

the second equation showing that F is positive or negative according as θ is acute or obtuse.

From the third of the above equations, we obtain

$$P = \frac{Wb\cos a}{(a+b)\sin(a+\theta)}.$$

Thus,

$$F = P\cos\theta = \frac{Wb\cos a\cos\theta}{(a+b)\sin(a+\theta)} = \frac{Wb}{(a+b)(\tan a+\tan\theta)}$$

Also,

$$R = W - P\sin\theta = W - \frac{Wb\cos a\sin\theta}{(a+b)\sin(a+\theta)} = W\left\{1 - \frac{b\tan\theta}{(a+b)(\tan a+\tan\theta)}\right\}$$

$$= W \cdot \frac{(a+b)\tan\alpha + a\tan\theta}{(a+b)(\tan\alpha+\tan\theta)}$$

$$\therefore \frac{F}{R} = \frac{b}{(a+b)\tan\alpha + a\tan\theta}.$$

Now, if θ is acute, $\frac{F}{R}$ must be $< \mu$;

$$\therefore\ b < \mu(a+b)\tan\alpha + \mu a\tan\theta\ ;$$

$$\therefore\ \tan\theta > \frac{b-\mu(a+b)\tan\alpha}{\mu a}.$$

And, if θ is obtuse, $-\frac{F}{R}$ must be $< \mu$;

$$i.e.\ \frac{b}{-(a+b)\tan\alpha + a(-\tan\theta)} \text{ must be } < \mu\ ;$$

$$\therefore\ b < -\mu(a+b)\tan\alpha + \mu a(-\tan\theta)\ ;$$

$$\therefore\ (-\tan\theta) > \frac{b+\mu(a+b)\tan\alpha}{\mu a}.$$

Thus we see that θ must lie between the two angles

$$\tan^{-1}\left(-\frac{a+b}{a}\tan\alpha \pm \frac{b}{\mu a}\right).$$

*** 212. Equilibrium of a Rigid Body resting at Two Points against Rough Surfaces.**—Let a rigid body rest in equilibrium against two rough surfaces at A and B, the area in contact being in each case so small that it may be treated as a point ; and suppose that, in addition to the resistances of the surfaces, a given system of forces acts upon the body in a plane containing the points A and B.

Let λ and λ' be the angles of friction, and AM and BN the normals, at A and B respectively. Draw AK and AK', each making an angle λ with AM, also BL and BL', each making an angle λ' with BN, so that the four lines form the quadrilateral $KLK'L'$.

The resultant resistance at A must act in a direction intermediate to AK and AK', and may be of any magnitude ; also, the resultant resistance at B must act in a direction intermediate to BL and BL', and may be of any magnitude.

Thus the lines of action of the resultant resistances at A and B meet at some point within the quadrilateral $KLK'L'$,

and are together equivalent to a single resultant force acting through this point and intermediate in direction to AK and BK.

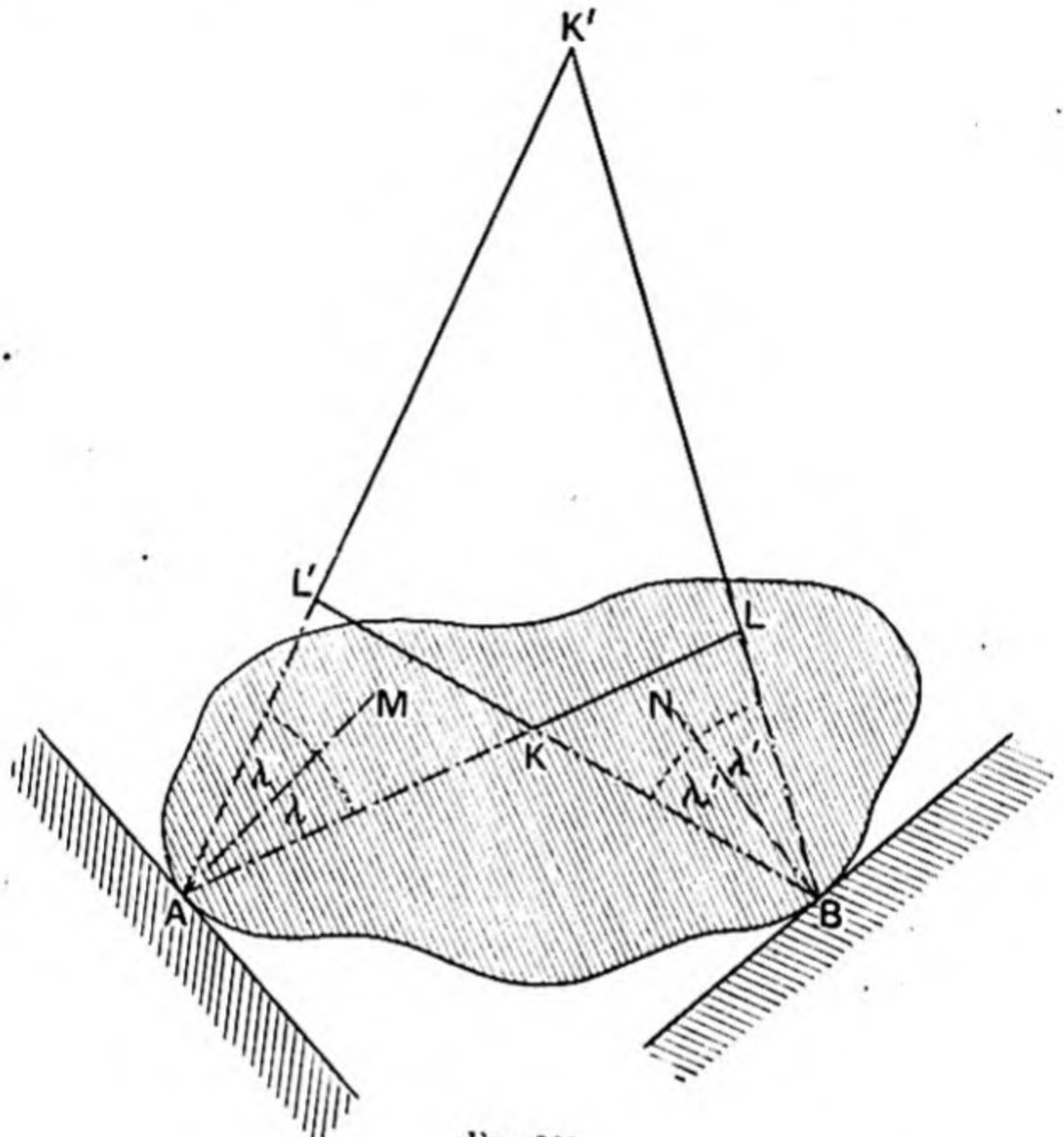


FIG. 185.

Hence, for equilibrium, it is necessary and sufficient that the given system of forces should reduce to a resultant whose line of action intersects the quadrilateral $KLK'L'$, and whose direction is intermediate to KA and KB.

*** 213. Equilibrium of a Ladder resting against a Wall.**—Suppose that a ladder AB, regarded as a fine straight rigid rod, rests against a rough horizontal plane at A, and against a rough vertical wall at B, the vertical plane containing the ladder being perpendicular to the wall.

Let w be the weight of the ladder and G its centre of gravity; also let λ and λ' be the angles of friction at A and B respectively. Draw AK and AK' each making an angle λ with the upward vertical at A, and draw BL and BL' each making an angle λ' with the horizontal drawn away from the wall at B.

If the inclination of the ladder to the *vertical* is less than λ, it will rest without danger of slipping however it is loaded; for the resultant weight will act downwards along some vertical line which intersects the area $L'BK'$, and all parts of this area lie within each of the angles KAK' and LBL'.

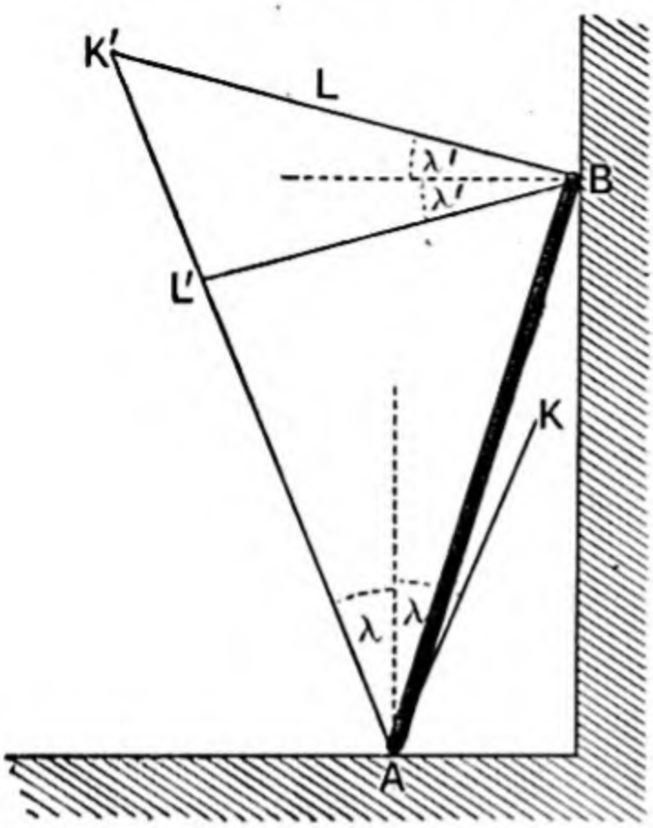


FIG. 186.

If the inclination of the ladder to the *vertical* is greater than λ, the four lines AK, AK', BL, BL' form a quadrilateral $KLK'L'$. Then, when the ladder is loaded in any manner, it is necessary and sufficient for equilibrium that the line of action of the resultant weight should intersect the area $KLK'L'$.

Let the vertical through L meet AB in H. Then we see that it is necessary and sufficient for equilibrium that the resultant weight should act between A and H.

If the resultant weight acts along the line LH, the equilibrium is limiting, and the actions at A and B are then along AL and BL respectively, and are determinate. In other cases these resistances cannot be found.

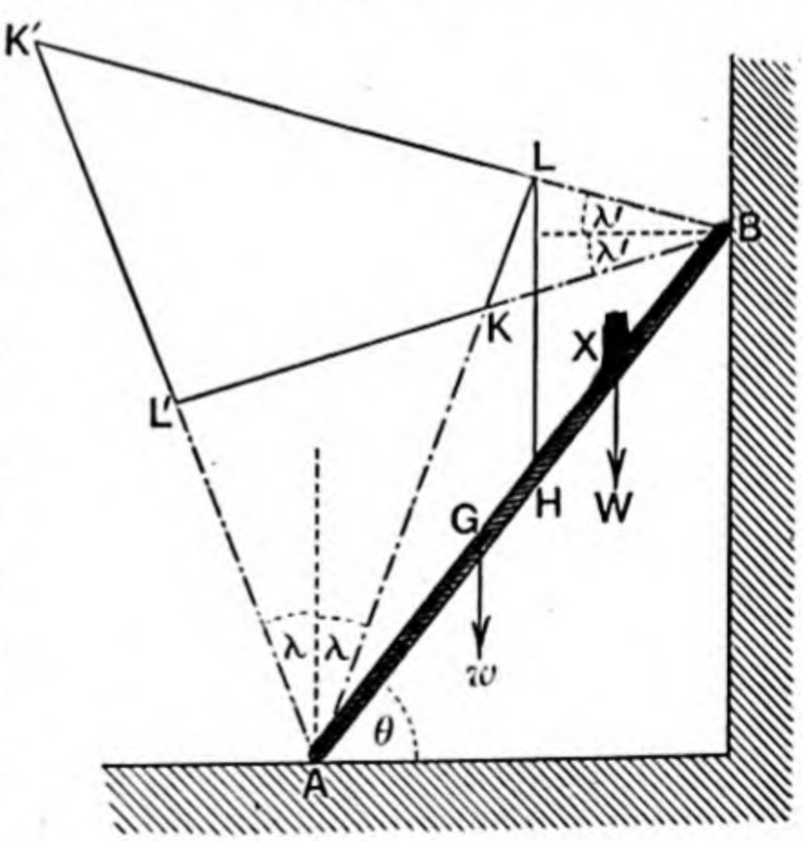


FIG. 187.

Let θ be the inclination of the ladder to the horizontal, and suppose that it rests without slipping when there is no load upon it. Then it is necessary and sufficient that AG be $< AH$.

Let a and b be the lengths of AG and BG respectively,

and let μ and μ' be the coefficients of friction at A and B respectively.

Then

$$\frac{AH}{AG} = \frac{AH}{AL}\cdot\frac{AL}{AB}\cdot\frac{AB}{AG}$$
$$= \frac{\sin ALH}{\sin AHL}\cdot\frac{\sin ABL}{\sin ALB}\cdot\frac{a+b}{a}$$
$$= \frac{\sin\lambda}{\cos\theta}\cdot\frac{\sin(\theta+\lambda')}{\cos(\lambda-\lambda')}\cdot\frac{a+b}{a}.$$

Hence, we must have

$$(a+b)\sin\lambda\sin(\theta+\lambda') > a\cos(\lambda-\lambda')\cos\theta,$$
$$\therefore (a+b)\sin\lambda\cos\lambda'\sin\theta > \{a\cos(\lambda-\lambda') - (a+b)\sin\lambda\sin\lambda'\}\cos\theta,$$
i.e. $$(a+b)\sin\lambda\cos\lambda'\sin\theta > (a\cos\lambda\cos\lambda' - b\sin\lambda\sin\lambda')\cos\theta,$$
$$\therefore (a+b)\mu\sin\theta > (a - b\mu\mu')\cos\theta,$$
$$\therefore \theta > \theta_1, \text{ where } \tan\theta_1 = \frac{a-b\mu\mu'}{(a+b)\mu}.$$

Now let θ be $> \theta_1$, and $< \frac{\pi}{2} - \lambda$, *i.e.* let $\tan\theta$ be $> \frac{a-b\mu\mu'}{(a+b)\mu}$ and $< \frac{1}{\mu}$, so that the ladder rests without slipping when unloaded and H lies between A and B. Then, it is clear that if any body of weight W is placed on the ladder at a point between A and H, the equilibrium will not be disturbed. If, however, the body is placed at a point X between H and B, the equilibrium will, or will not, be disturbed, according as the resultant of w and W acts above or below H.

Let the distance AX be of length x. Then, for the ladder to be in equilibrium when loaded, we must have

$$\frac{Wx + wa}{W + w} < AH,$$
$$\therefore Wx + wa < (W+w)\frac{(a+b)\sin\lambda\sin(\theta+\lambda')}{\cos\theta\,.\cos(\lambda-\lambda')},$$
i.e. $$Wx + wa < (W+w)\frac{(a+b)\mu(\tan\theta+\mu')}{1+\mu\mu'}.$$

Now the greatest possible value of x is $a+b$. Hence, in order that the equilibrium may not be disturbed for any position of the load, we must have

$$W(a+b) + wa < (W+w)\cdot\frac{(a+b)\mu(\tan\theta+\mu')}{1+\mu\mu'},$$

$$\therefore\ W(a+b)\{(1+\mu\mu')-\mu(\tan\theta+\mu')\} < w\{(a+b)\mu(\tan\theta+\mu')-a(1+\mu\mu')\},$$

i.e. $$W(a+b)(1-\mu\tan\theta) < w\{\mu(a+b)\tan\theta - a + b\mu\mu'\},$$

$$\therefore\ W < w\,.\,\frac{\tan\theta - \dfrac{a-b\mu\mu'}{\mu(a+b)}}{\dfrac{1}{\mu}-\tan\theta},$$

i.e. $$W < w\,.\,\frac{\tan\theta-\tan\theta_1}{\cot\lambda-\tan\theta}.$$

* **214.** Ex. 41.—*Determine the inclination of the ladder to the horizontal when, being unloaded, it is on the point of slipping.*

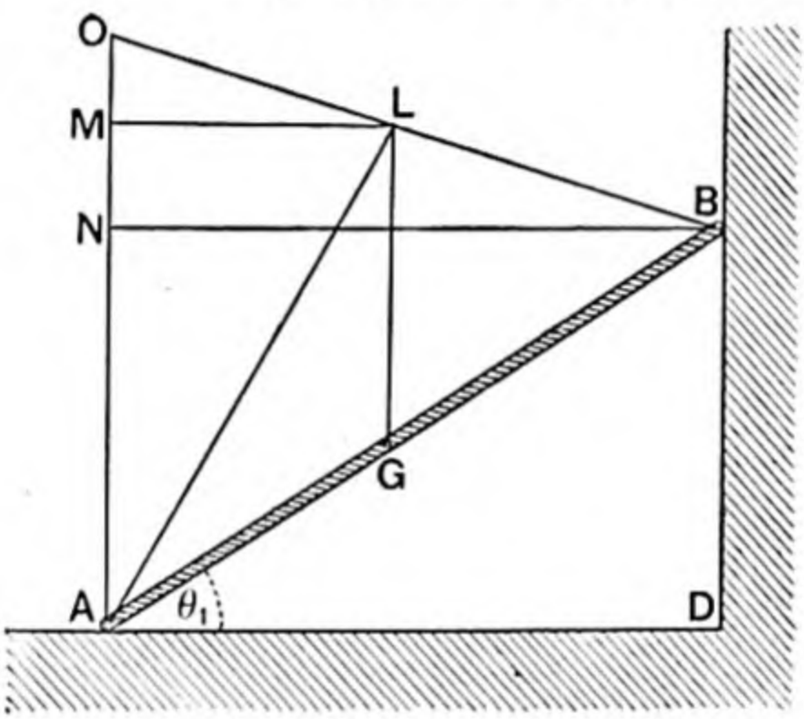


FIG. 188.

(i) *Geometrically.*—As the ladder is in *limiting* equilibrium, the centre of gravity G lies vertically below the point L.

Let the vertical through A meet the horizontals through L and B in M and N respectively, and BL produced in O. Then, as MAL and MLO are the angles of friction for the ends A and B respectively, we have

$$ML = \mu\,.\,MA \text{ and } MO = \mu'\,.\,ML = \mu\mu'\,.\,MA.$$

Also $$OL : LB = AG : GB = a : b.$$

Now $$MN : MO = LB : LO = b : a,$$

$$\therefore\ MN \text{ is of length } \frac{b}{a}\,.\,\mu\mu'\,.\,MA.$$

And $$NB : ML = OB : OL = a+b : a,$$

$$\therefore\ NB \text{ is of length } \frac{a+b}{a}\,.\,\mu\,.\,MA.$$

Thus, $$\tan\theta_1 = \frac{MA - MN}{NB} = \frac{1 - \frac{b}{a}\mu\mu'}{\frac{a+b}{a}\mu}$$

$$= \frac{a - b\mu\mu'}{(a+b)\mu}.$$

[For a method of determining the limiting position of the ladder by geometrical construction, see the Author's *Elementary Geometrical Statics.*]

(ii) *Analytically.*—Let R and R' be the normal reactions at A and B respectively; then, as the equilibrium is *limiting*, the frictions at those points are as indicated in Fig. 189.

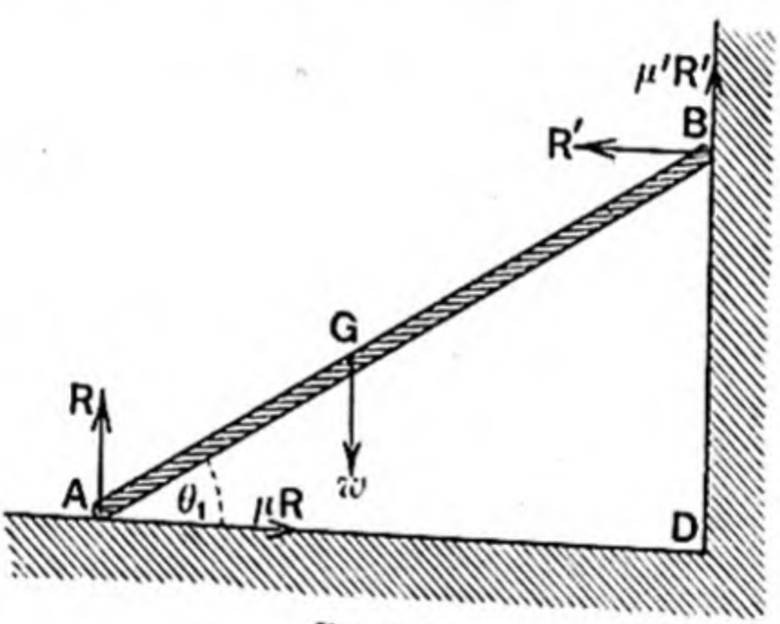


FIG. 189.

Resolving vertically and horizontally, we have

$$\begin{cases} R + \mu' R' = w \\ \mu R = R' \end{cases}$$

∴ eliminating R,

$$R'(1 + \mu\mu') = \mu w.$$

Also, taking moments about A, we have

$$w \,.\, a\cos\theta_1 = \mu' R' \,.\, (a+b)\cos\theta_1 + R' \,.\, (a+b)\sin\theta_1.$$

∴ substituting for R'

$$wa\cos\theta_1 = \frac{\mu w(a+b)}{1+\mu\mu'}(\mu'\cos\theta_1 + \sin\theta_1).$$

$$\therefore\ a(1+\mu\mu') = \mu(a+b)(\mu' + \tan\theta_1).$$

$$\therefore\ \mu(a+b)\tan\theta_1 = a(1+\mu\mu') - \mu\mu'(a+b).$$

$$\therefore\ \tan\theta_1 = \frac{a - b\mu\mu'}{(a+b)\mu}.$$

* **215.**—Ex. 42.—*The angle θ being less than θ_1, the ladder is prevented from slipping by means of a horizontal force applied at A. Compare the force necessary to move the lower end of the ladder towards the wall with the force necessary to prevent it from slipping away from the wall.*

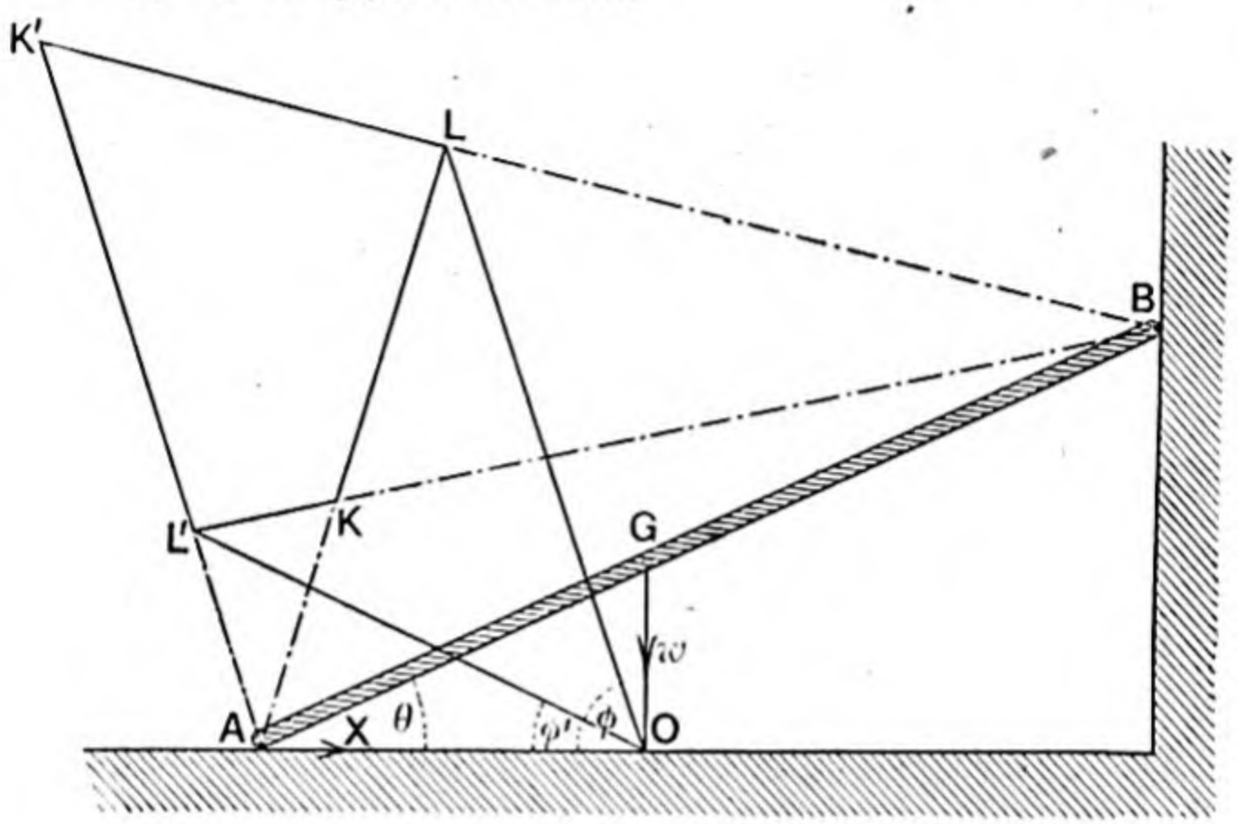


FIG. 190.

(i) *Geometrically.*—Let the vertical through the centre of gravity G meet the horizontal through A in O, and suppose that the ladder is at rest when a force X is applied at A in the direction AO.

The line of action of the resultant of X and w passes through O. For equilibrium, it is necessary and sufficient that this line should intersect the area $KLK'L'$. Thus the line of action of the resultant of X and w must lie between the positions LO and $L'O$. If it is beyond these limiting positions, the ladder slips; if it is along LO, the ladder is on the point of slipping *away from* the wall; if along $L'O$, the ladder is on the point of slipping *towards* the wall. Thus, if LO and $L'O$ make angles ϕ and ϕ' with the horizontal, the force X must be intermediate in value between $w \cot \phi$ and $w \cot \phi'$.

It will be an interesting exercise for the student to prove from the figure that

$$\cot \phi = \frac{a - b\mu\mu' - \mu(a+b)\tan\theta}{(a+b)(\tan\theta + \mu')}$$

and

$$\cot \phi' = \frac{a - b\mu\mu' + \mu(a+b)\tan\theta}{(a+b)(\tan\theta - \mu')}.$$

We proceed to the analytical solution of the question.

(ii) *Analytically.*—Suppose that the lower end of the ladder is on the point of slipping *away from* the wall when the horizontal force applied at A has the value X_1. Then the friction at A is towards the wall and the friction at B is upwards, so that, if R and R' are the normal reactions at A and B respectively, then the forces acting on the ladder are as indicated in Fig. 191.

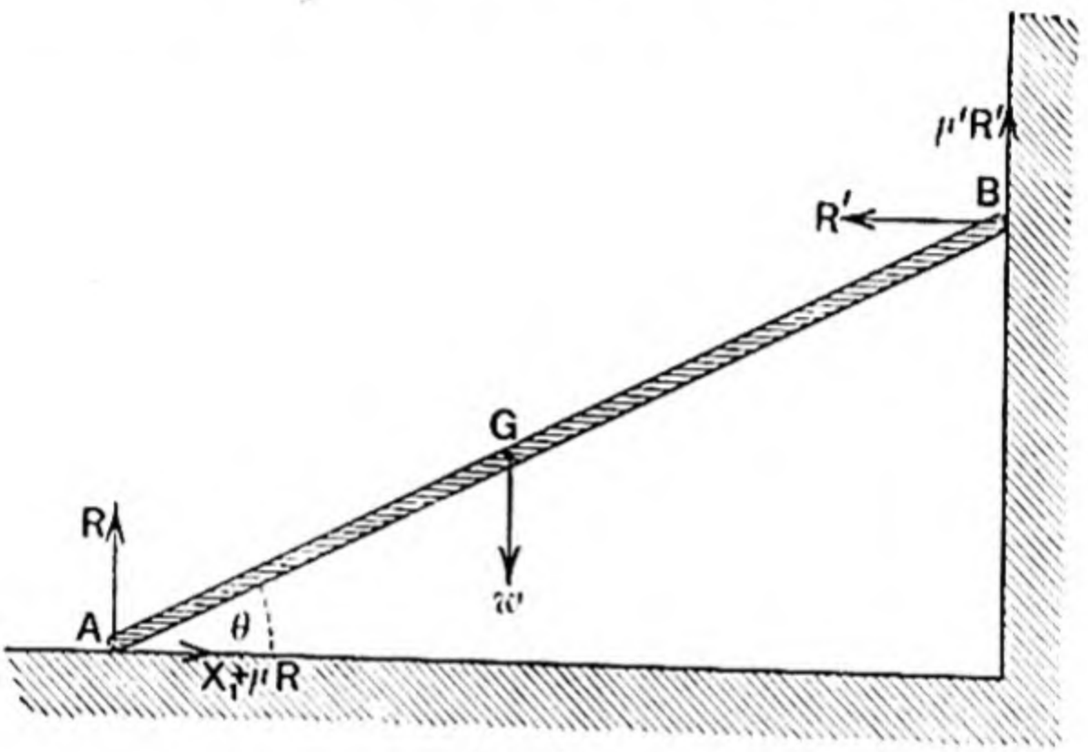


FIG. 191.

Resolving vertically and horizontally, we have

$$\begin{cases} R + \mu' R' = w \\ \mu R + X_1 = R' \end{cases}$$

$\therefore$ eliminating R,

$$R'(1 + \mu\mu') = \mu w + X_1.$$

Also, taking moments about A, we have

$$w \,.\, a \cos\theta = \mu' R' \,.\, (a + b)\cos\theta + R' \,.\, (a + b)\sin\theta.$$

$\therefore$ substituting for R',

$$wa\cos\theta = \frac{(\mu w + X_1)(a + b)(\mu' \cos\theta + \sin\theta)}{1 + \mu\mu'}$$

$$\therefore\ a(1 + \mu\mu')w = (\mu w + X_1)(a + b)(\mu' + \tan\theta).$$

$$\therefore\ X_1 = \frac{a - b\mu\mu' - \mu(a + b)\tan\theta}{(a + b)(\mu' + \tan\theta)} w.$$

$$= \frac{\mu(\tan\theta_1 - \tan\theta)}{\tan\theta + \mu'} w.$$

Suppose that the lower end of the ladder is on the point of slipping *towards* the wall when the horizontal force applied at A has the value X_2. Then we can deduce X_2 from X_1 by changing the signs of μ and μ'.

Thus,
$$X_2 = \frac{a - b\mu\mu' + \mu(a+b)\tan\theta}{(a+b)(\tan\theta - \mu')} w$$
$$= \frac{\mu(\tan\theta_1 + \tan\theta)}{\tan\theta - \mu'} w.$$

Hence,
$$\frac{X_2}{X_1} = \frac{\tan\theta_1 + \tan\theta}{\tan\theta_1 - \tan\theta} \cdot \frac{\tan\theta + \mu'}{\tan\theta - \mu'}.$$
$$= \frac{\tan\theta_1 + \tan\theta}{\tan\theta_1 - \tan\theta} \cdot \frac{\tan\theta + \tan\lambda'}{\tan\theta - \tan\lambda'}.$$

* EXAMPLES XXV

1. A uniform beam AB, of length $2a$ and weight W, is supported in a horizontal position, with its extremity A against a rough vertical wall, by means of a force applied to it at C in a direction inclined at an angle α with CA. If CA is of length b, show that the applied force must be of magnitude $\frac{a}{b} W \operatorname{cosec}\alpha$, and that the coefficient of friction at A must be greater than $\frac{a-b}{a}\tan\alpha$.

2. A uniform beam, of weight W, laid on a horizontal plane, can be just moved in its own direction by pushing it with a horizontal force of magnitude P less than W. Find the least force which can move it in its own direction, and determine the direction of this force.

3. If the beam of the preceding question is pulled by a gradually increasing force applied by a fine string attached at one end A, determine the least inclination of the string to the horizontal, in order that A may be raised from the ground.

4. A rigid uniform bar, of weight W, rests with its lower end in contact with a horizontal rough plane and makes an angle of 60° with the horizontal, being kept in position by a fine horizontal thread attached to its upper end. Find the tension of the thread, and the magnitude and direction of the force exerted at the lower end.

5. In the preceding question, if the coefficient of friction at the lower end is $\frac{1}{3}$, find the weight of the greatest mass which can be suspended from the upper end of the bar, without disturbing the equilibrium.

6. A uniform beam AB, whose length is $12\frac{1}{2}$ feet, rests with one extremity A on a rough horizontal plane AC, and is kept from falling forwards by a fine cord BC, 20 feet long, whose extremity is attached to a fixed point C in the plane, directly behind the beam. If the beam is on the point of slipping when $AC = AB$, find the coefficient of friction.

7. A uniform beam AB, of weight W, is supported in a position inclined to the horizontal, with one end B resting on a rough horizontal plane, by means of a force of magnitude $\frac{2}{5}W$ acting along AC, such that B and C are on the same side of the vertical through A, and AC is in an upward direction inclined at an angle of 30° to the horizontal. Find the inclination of the beam to the horizontal and the magnitude of the resultant resistance at B, and show that the coefficient of friction is greater than $\frac{1}{4}\sqrt{3}$.

8. In the preceding example, if the coefficient of friction is $\frac{1}{2}$, show that the beam can be supported in the same position as before, by a force of sufficient magnitude applied at A in any direction between the two directions which make angles of 15° and 105° with the horizontal.

9. A rigid uniform bar is supported, with its lower extremity resting against a rough plane of inclination α, by means of a force applied to its upper extremity in the direction of the lines of greatest slope of the plane. Prove that, if the rod makes an angle $\alpha+\beta$ with the vertical, and an angle β ($<\alpha$) with the normal to the plane, then the coefficient of friction cannot be less than $\frac{1}{2}\sec\alpha\sec\beta\sin(\alpha-\beta)$.

10. A uniform ladder of length l and weight w is being gradually tilted off the ground by a man who can reach up to a height h. If the ground is perfectly smooth where the lower end touches it, and if $l>2h$, prove that the ladder cannot be raised into a vertical position unless its lower end is weighted, and find the least weight that will suffice. If the ground is rough, show by a diagram that as soon as the inclination of the ladder to the vertical is less than the angle of friction, the ladder may be supported at a point below its middle point, provided that a force of suitable magnitude and direction is applied to the point of support, and show how to determine this force when equilibrium is limiting, having given its point of application.

11. A uniform beam AB, of weight W, rests with one end B upon a rough horizontal plane, and has its other end A attached to a light string, which passes over a smooth pulley E, situated vertically over the beam, and supports a mass of weight P. Show that, if the angle of friction between the beam and the plane is λ, then the inclination θ of AE to the horizontal must satisfy the condition

$$\theta-\lambda>\cos^{-1}\left(\frac{W}{P}\sin\lambda\right),$$

and the tangent of the inclination of the beam to the horizontal is

$$\tfrac{1}{2}\left(\frac{W}{P}\right)\sec\theta-\tan\theta \text{ and is } >\tfrac{1}{2}(\cot\lambda-\tan\theta).$$

12. A fine light rod rests in a horizontal position between two rough inclined planes, the vertical plane which contains the rod being at right angles to each of the inclined planes. Show how to determine, by geometrical construction, that portion of the rod, at any point of which a small heavy ring can be placed without causing the rod to slip.

13. A rigid uniform bar, of weight W, is placed with one end A resting on a rough horizontal plane and the other end B against a smooth vertical wall, the vertical plane containing the bar being perpendicular to the wall. Supposing the bar to be on the point of

motion, and the coefficient of friction to be $\frac{1}{2}$, determine the magnitudes of the reactions at A and B, and the inclination of the bar to the horizontal.

14. A ladder rests against a rough horizontal plane and against a smooth vertical wall, the vertical plane containing the ladder being perpendicular to the wall. The foot of the ladder is prevented from slipping *away from* the wall by means of a horizontal force applied to it in a direction towards the wall. Show that, in order to move the ladder *towards* the wall, the horizontal force must be increased by an amount not less than $2\mu W$, where W is the weight of the ladder and μ the coefficient of friction between the ladder and the horizontal plane.

15. A heavy circular wheel rests on a rough plane inclined at an angle α to the horizontal, being supported by a fine string wrapped round its circumference and attached to a point in the plane. Find the inclination of the straight portions of the string to the horizontal when the wheel is just on the point of slipping, supposing that the plane of the wheel is vertical and perpendicular to the inclined plane, that its centre of gravity is at its centre, and that the coefficient of friction between the wheel and the plane is μ. Show also that there are two positions of limiting equilibrium.

16. A heavy circular wheel is suspended in a vertical plane from a point B in a rough vertical wall, by means of a smooth wire of inappreciable weight through its centre O, and rests at right angles to the wall and touching it at the point A. A fine string is fixed to the rim of the wheel and passes over it on the side away from A. If the inclination α of the wire OB to the horizontal is less than 2λ, where λ is the angle of friction at A, show that no force, however great, applied along the string will turn the wheel unless the direction of the string makes with BO produced an angle less than

$$\sin^{-1}\frac{\sin(\alpha-\lambda)}{\sin\lambda}.$$

(Positive angles are measured from OA to OB.)

17. A rigid uniform bar AB, of weight w, rests in a horizontal position at right angles to a vertical wall AD, being supported by a fine string CD connecting the middle point C of the bar with a point D of the wall situated vertically above A. A mass of weight W is suspended from a point E of the bar. Show that, if CD is inclined to the horizontal at an angle α greater than $\tan^{-1}\left(\frac{2W+w}{W}\mu\right)$, where μ is the coefficient of friction at A, then the point E must be situated in a certain portion of the bar of length equal to

$$\frac{W+w}{W}\cdot\frac{\mu\tan\alpha}{\tan^2\alpha-\mu^2}$$

of the length of the bar.

18. C and D are two small rough pegs, situated one foot apart in a straight line inclined to the horizontal at an angle of 60°. A uniform rod $ACDB$, weighing 10 pounds and of length 3 feet, is placed in this line, under C and above D, so that $AC=BD=1$ foot. Find the least mass which, suspended from the *upper* end B, will keep the

rod in the above position, the angle of friction between the rod and each peg being 45°.

19. A and B are two small rough pegs, situated at a distance c apart in a straight line inclined to the horizontal at an angle α, and B being at a higher level than A. A heavy rod rests on B and passes under A, the angle of friction at each peg being of magnitude λ. Show that (i) if α is $<\lambda$, the rod will rest so long as its centre of gravity is anywhere above B; (ii) if α is $>\lambda$, equilibrium will exist so long as the centre of gravity of the rod is at a distance above B greater than $\dfrac{c\sin(\alpha-\lambda)}{2\cos\alpha\sin\lambda}$.

20. A lamina, whose centre of gravity is at G, rests vertically upon a rough inclined plane, which it touches at two points A and B situated in a line of greatest slope, so that B is at a higher level than A. GN is the perpendicular from G upon AB, and the straight lines AN, NB, GN are of lengths a, b, h respectively. The inclination of the plane to the horizontal is gradually increased until the lamina slips. Show that, if μ and μ' are the coefficients of friction at A and B respectively, then a must be greater than μh, and the limiting inclination of the plane to the horizontal is

$$\tan^{-1}\frac{\mu' a+\mu b}{a+b-h(\mu-\mu')}.$$

21. A uniform ladder, of weight W, rests against a rough horizontal plane and against a rough vertical wall, the vertical plane containing the ladder being perpendicular to the wall. If the coefficient of friction at each end is $\frac{1}{3}$, find

(i) The limiting inclination of the ladder to the horizontal when unloaded.

(ii) The limiting inclination of the ladder to the horizontal when a man of weight W stands at the top.

(iii) The weight of the heaviest man who can ascend the ladder when it is inclined to the horizontal at an angle $\tan^{-1}2$.

(iv) The weight of the greatest load which can be placed on the ladder three-quarters of the way up when it is inclined to the horizontal at an angle $\tan^{-1}2$.

(v) The greatest height to which a man of weight W can ascend when the ladder is inclined to the horizontal at an angle $\tan^{-1}2$.

22. A ladder AB, 15 feet long, rests against the ground at A and against a rough vertical wall at B, the coefficients of friction at A and B being $\frac{2}{3}$ and $\frac{1}{2}$ respectively; the centre of gravity, G, is 6 feet from A: find the inclination to the horizontal at which the ladder will be just about to slip.

23. If the ladder of the preceding question is placed so that the distance of A from the wall is twice as great as the distance of B above the ground, and a boy, whose mass is one-fifth of that of the ladder, ascends it in this position, how far will he be able to go before the ladder begins to slip?

24. A ladder AB rests against a rough horizontal plane at A, and against a rough vertical wall at B, the vertical plane containing the

ladder being perpendicular to the wall. It is on the point of slipping when inclined to the horizontal at an angle of 45°. If the coefficients of friction at A and B are $\frac{3}{8}$ and $\frac{1}{3}$ respectively, find the position of the centre of gravity of the ladder.

25. A uniform ladder rests against a rough horizontal plane at A, and against a rough vertical wall at B, the vertical plane containing the ladder being perpendicular to the wall. It is just on the point of slipping when inclined at an angle of 45° to the horizontal, and then the least horizontal force which, applied to the foot of the ladder, will move it towards the wall is $\frac{3}{5}$ths of the weight of the ladder. Find the coefficients of friction at A and B.

26. If the inclination of a ladder to the horizontal ground is less than the angle of friction between the ladder and the vertical wall, show that no horizontal force, however great, applied to the foot of the ladder can move it towards the wall.

27. A uniform ladder rests with one end against a vertical wall and the other on a horizontal ground, the vertical plane containing the ladder being perpendicular to the wall. If the inclination of the ladder to the wall is 45°, compare the least horizontal forces which, applied to the foot of the ladder, will move it towards the wall or prevent it from slipping away from the wall, the coefficient of friction being $\frac{1}{4}$ for each end of the ladder.

28. A ladder is placed with one end on a rough horizontal plane, and the other against a rough vertical wall; show how to determine by geometrical construction the limiting position of equilibrium, having given the coefficients of friction and the centre of gravity of the ladder.

If an additional load is placed at any point on the ladder, in this limiting position, find whether the equilibrium will be disturbed or not.

29. A uniform ladder rests with one end against a vertical wall and the other on the horizontal ground, the vertical plane containing the ladder being perpendicular to the wall. If the wall and the ground are equally rough, show that the limiting inclination of the ladder to the *vertical* is twice the angle of friction.

30. In Art. 213, if $a < \mu\mu' b$, show that the ladder will rest at any inclination to the wall.

31. The ladder of Art. 213 rests in limiting equilibrium against a rough vertical wall and rough horizontal ground. Show that a man of weight W can ascend to the top of the ladder while in this position, provided that a man of weight not less than $\frac{b}{a}W$ stands on the ladder at the bottom.

32. A uniform lamina, in the form of an equilateral triangle, rests within a rough, hollow, right circular cylinder whose axis is horizontal, the plane of the lamina being at right angles to the axis of the cylinder, and the internal radius of the cylinder being equal to a side of the triangle. Show that, in the limiting position of equilibrium, the base of the triangle is inclined to the horizontal at an angle equal to twice the angle of friction.

33. A uniform rod AB rests within a rough, hollow, right circular cylinder whose axis is horizontal, the vertical plane which contains the rod meeting the axis of the cylinder at right angles in the point O. If AB subtends an angle α at O, and if λ is the angle of friction at each end, show that the limiting inclination of the rod to the horizontal is $\tan^{-1}\left(\dfrac{\sin 2\lambda}{\cos\alpha + \cos 2\lambda}\right)$.

34. A uniform body of weight w is in the form of two right circular cylinders, of radii a and b, fastened together with their axes in a straight line. The larger cylinder, of radius a, fits in a rough hollow cylinder of slightly larger radius, fixed with its axis horizontal. A light cord is coiled round the smaller cylinder of radius b, and supports at its extremity a mass of weight W. Show that, in order that the body may rest in the hole without slipping, b must be less than $a\left(\dfrac{W+w}{W}\right)\sin\lambda$, where λ is the angle of friction between the surfaces in contact.

* MISCELLANEOUS EXAMPLES

1. Assuming the parallelogram law for the direction of the resultant of two forces acting at a point, and without assuming the law true for magnitude, prove that if forces, acting at a point, are represented by the sides of a quadrilateral taken one way round, then they are in equilibrium.

2. Of two forces P and Q, the magnitude of Q is given and also a point O on its line of action. Show that if the forces are equivalent to a given force R, the envelope of the lines along which it is possible for P to act is a conic, of which O is the focus and the line of action of R the corresponding directrix.

3. A rigid uniform rod is supported by two light strings attached to its extremities and to two fixed points situated at the same level. Prove that, if the strings make equal angles with the vertical, then they must be equal in length.

4. A circular plate, of radius a, is fixed in a vertical plane. Upon its smooth rim (and in its plane) are placed two uniform rods AB and AC freely jointed together at A, each being of length $2b$ and weight W. Find an equation for θ, the angle which each rod makes with the vertical in the position of equilibrium, and express in terms of W and θ the pressure upon the plate and the action at the joint.

5. A heavy particle hangs by a light string which is drawn aside so that it may rest inclined at an angle to the vertical. The particle is held in this position by means of a horizontally applied force, which after a time is suddenly withdrawn. If the tensions of the string before the withdrawal and immediately afterwards are of magnitudes S and T respectively, show that the particle is of weight $\sqrt{ST}$.

6. If $ABCDEF$ is a regular hexagon, find the resultant of forces represented by the lines AC, AD, AE.

7. Four straight lines in a plane intersect, two at a time, in the six points A, B, C, D, E, F, and O is any other point in the same plane. Give a geometrical construction for the resultant of the six forces represented by OA, OB, OC, OD, EO, FO.

8. a, b, c, d are the middle points of the sides of a plane quadrilateral $ABCD$. Show that, if O is any point in its plane, then the system of forces represented by Oa, Ob, Oc, Od is equivalent to the system represented by OA, OB, OC, OD.

9. $ABCDE$ is a regular pentagon, and forces acting at a point are represented in magnitude and direction by the lines AB, AC, AD, AE, BC, BD, BE, CD, CE, DE; prove that the resultant is in the direction AE and find its magnitude.

10. Forces $P_1, P_2, \ldots, P_n$ acting in one plane at a point O are in equilibrium. Any transversal cuts their lines of action in the points $L_1, L_2, \ldots, L_n$ respectively; and the lengths $OL_1, OL_2, \ldots, OL_n$ are positive when the direction from O to L is in the same direction as that of P. Prove that $\Sigma\left(\frac{P}{OL}\right)=0$.

11. A fine light string, of length l, has four particles each of weight W attached symmetrically at distances $\frac{1}{5}l$ apart, and is in equilibrium with its ends fastened at two points fixed in the same horizontal line. If the distance a between the points of support is very little less than l, prove that the horizontal tension is nearly

$$W\left(\frac{l}{l-a}\right)^{\frac{1}{2}}.$$

12. O is the orthocentre of a triangle ABC. Forces are represented in magnitude and direction by OA, OB, OC. Prove that their resultant is represented in magnitude and direction by $2(OS)$, where S is the circum-centre of the triangle ABC.

13. D is a point in the plane of a triangle ABC, and I the centre of its inscribed circle. Show that the resultant of forces represented by $a(AD)$, $b(BD)$, $c(CD)$ is a force represented by $(a+b+c)ID$, where a, b, c are the lengths of the sides of the triangle ABC.

14. Four centres of force at the angles of a rectangle $ABA'B'$ repel a particle P, free to move along a circular wire circumscribing the rectangle, with the forces $\frac{\mu}{A'P}$, $\frac{\mu}{B'P}$, $\frac{\mu}{AP}$, $\frac{\mu}{BP}$ respectively. Prove that the particle will be at rest anywhere on the wire; and show that the resultant force is $\frac{\mu}{PM}+\frac{\mu}{PN}$, where M and N are the feet of the perpendiculars from A and B upon the tangent to the circle at P.

15. Four particles, knotted at different points on an endless fine string, repel one another with forces which are the same at equal distances, but which vary as the distance. Show that, when there is equilibrium, two opposite sides of the quadrilateral formed by the string must be parallel, and that the tensions along the parallel parts of the string are equal.

16. A uniform square lamina is supported, so that its plane is vertical and one of its corners is in contact with a fixed smooth plane of inclination α, by a force applied at the opposite corner in a direction parallel to the line of slope of the inclined plane. Prove that the diagonal which passes through the point of contact makes with the vertical an angle 2α.

17. A smooth pencil, whose cross-section is a circle of radius r, is laid upon a flat and smooth ruler, whose breadth is $2c$, so that its axis is parallel to the edge of the ruler; the two are now bound

together by a fine string, which, passing under the ruler and over the pencil, lies in a plane perpendicular to the axis of the pencil; show that the pressure between the pencil and the ruler is $\left(\frac{4rc}{r^2+c^2}\right)T$, where T is the tension of the string.

18. A body of weight W is attached to a fine thread, and the ends of the thread are attached to small rings, of weights P and Q, which can slide on a smooth vertical circular wire. Prove that, if those radii of the circle which pass through the rings make angles α and β with the vertical, and the portions of the thread make angles γ and δ with the vertical, then

$$\frac{P}{\cot\alpha-\cot\gamma}=\frac{Q}{\cot\beta-\cot\delta}=\frac{W}{\cot\gamma+\cot\delta}.$$

19. A uniform rod of length l rests in equilibrium, partly within and partly outside a smooth hemispherical bowl of radius a, whose rim is horizontal. Show that the inclination θ of the rod to the horizontal is determined by

$$2AD = l\cos\theta = 4a\cos 2\theta.$$

Show also that l must be less than $4a$ and greater than $\frac{8}{3}a\sqrt{6}$.

20. A small ring is capable of motion along a smooth wire of circular shape, and is attracted by forces, varying as the distance and of equal absolute intensities, towards two given external points in the plane of the circle. Show that, in any position of the ring, the resultant attraction passes through a fixed point; and give a geometrical construction for a position of equilibrium.

21. Three small rings A, B, C, of unequal masses, can slide freely on a smooth fixed circular wire, and mutually repel each other, the force between any two being measured by the product of their masses and their distance apart. Show that, when there is equilibrium, the masses are proportional to the sides of the pedal triangle DEF, formed by joining the feet of the perpendiculars from A, B, C upon the opposite sides. Hence find a geometrical construction for the triangle ABC when the masses are given. Show also that the pressures on the circle are proportional to the masses.

22. Three forces, acting at a point, can be represented in magnitude and direction by the sides of a triangle taken one way round. Prove, from first principles, that the algebraic sum of the moments of the three forces about any point in their plane is zero.

23. Prove from statical considerations that the line joining the middle points of the diagonals of a quadrilateral described about a circle, passes through the centre of the circle.

24. A particle P is attracted towards two fixed points A and B, by forces represented by $\mu\,(PA)$ and $\mu'\,(PB)$. Find the locus of P such that the sum of the moments of the forces about a given point O may be equal to a given constant.

25. A uniform beam AB, of length l and weight W, is movable in a vertical plane about a smooth hinge at A. To the end B is attached a fine string which passes over a small smooth pulley, fixed at a height h $(>l)$ vertically over A, and carries at its other end a

mass of weight w. Show that the beam is in stable equilibrium when it is in a vertical position above A, provided $w > \left(\frac{h-l}{2h}\right) W$.

26. A pair of compasses of negligible mass, with a stiff hinge, lie open at an angle, and are to be closed by pressing the points towards each other. Show that they may be completely closed by a finite pressure if the legs are of equal length; but that if they are of unequal lengths a and b $(a > b)$, a pressure P can diminish the angle only until it is equal to $\sin^{-1}\frac{P_0}{P} - \sin^{-1}\frac{bP_0}{aP}$, where P_0 is the least possible pressure that could move the hinge, if open at the most favourable angle.

27. $ABCD$ is a quadrilateral. Four forces acting along AB, BD, DA, CA are represented by AB, BD, DA, CA respectively. Show that their resultant is represented in magnitude and direction by CA, and that its line of action intersects CD at a point P where $CP = QD$, BQ being drawn parallel to AD to meet CD in Q.

28. Any two points E and F are taken on the sides AB and CD of the parallelogram $ABCD$; G and H are the middle points of AC and EF respectively; prove that the resultant of forces acting along and represented by EC, ED, FA, FB is represented in magnitude and direction by $4(HG)$.

Prove also that the forces are equivalent to a force acting along HG and represented by $4(HG)$ together with a couple whose moment is the difference between the forces represented by AE and DF multiplied by the perpendicular from A upon CD.

29. The point O is the centre of the circle circumscribing the triangle ABC, and forces act along OA, OB, OC proportional respectively to the sides BC, CA, AB of the triangle; prove that their resultant passes through the centre of the inscribed circle.

30. Three forces acting at A are represented in magnitude and direction by AB, $2(CA)$, $3(BC)$; show that their resultant is represented by OA, where O is a point in BC such that B bisects OC.

31. Three forces act at the angular points of a triangle ABC, each parallel to the opposite side. If they have a resultant acting along the line joining the in-centre and orthocentre of the triangle, prove that they are proportional to

$$ab(a+b)\cos B - ac(a+c)\cos C$$

and the two similar expressions.

32. ABC is a triangle, and the line of action of a force P meets BC, CA, AB in D, E, F respectively. Show that, if the force is resolved into components acting along the sides of the triangle, then the component along BC is

$$\frac{BC \cdot AE \cdot AF}{EF \cdot AB \cdot AC} \cdot P.$$

33. Prove that, if E and F are the feet of the perpendiculars from two corners of a triangle ABC upon the opposite sides, then a force P acting along EF can be replaced by forces $P\cos A$, $P\cos B$, $P\cos C$ acting along the sides.

34. A particle at a point P, in the plane of a triangle ABC, is

acted upon by three forces towards the angular points of the triangle, that towards A being proportional to $PA \sin A \cos(B-C)$; there are two similar forces towards B and C. Show that the resultant force on P is directed towards and is proportional to its distance from the centre of the nine point circle of the triangle.

35. The circle inscribed in a triangle ABC touches the sides in the points D, E, F; prove that forces acting along the lines AD, BE, CF will be in equilibrium if they are in the ratios

$$BC.AD : CA.BE : AB.CF.$$

36. Under what conditions, geometrical or analytical, may four equal forces be in equilibrium which act all inwards, or all outwards, along the bisectors of the angles of a quadrilateral which may be inscribed in a circle?

37. $ABCD$ is a given quadrilateral in one plane; AE and AF are drawn parallel to BC and CD respectively, and any straight line is drawn meeting AB, AE, AF, AD respectively in P, Q, R, S. Prove that the resultant of forces acting on a rigid body at B, C, D, and represented in magnitude and direction by PQ, QR, RS respectively, is a force acting at A and represented by PS.

38. Forces P, Q, R, S act along the sides AB, BC, CD, DA of a quadrilateral, which is inscribed in a circle, whose centre is on AB. If they keep a rigid body in equilibrium, prove that

$$R^2 = P^2 + Q^2 + S^2 + \frac{2PQS}{R}.$$

39. Forces act along the sides AB, BC, CD, DA of a plane quadrilateral taken one way round, and their magnitudes are p, q, r, s times the lengths of the sides in which they act. Prove that they are equivalent to a couple if

$$\left.\begin{aligned}(p-q)OB &= (r-s)OD,\\ (q-r)OC &= (s-p)OA,\end{aligned}\right\}$$

where O is the intersection of AC and BD.

40. Forces act inwards perpendicular and proportional to the sides of a closed polygon; find the condition of equilibrium, and show that if the points at which they act divide the sides in the same ratio, they are the middle points of the sides.

41. A lamina can turn freely in its own plane about a fixed point in the plane. At different points of the lamina forces are applied which are constant in direction and magnitude. Show that, in general, there are two possible positions of equilibrium.

42. A ladder, of no appreciable weight, rests against a uniform smooth cube of weight W, standing on smooth ground with the foot of the ladder tied to the middle point of one of the lowest edges of the cube. A man of weight w ascends the ladder. Prove that, if the ladder projects above the top of the cube, the cube will tilt before he reaches the top of the cube unless

$$W > 2w \cos\alpha(\sin\alpha - \cos\alpha),$$

where α is the inclination of the ladder to the horizontal.

43. A uniform rod AB, of weight W, can turn freely round one end A; a fine cord is attached to a point C vertically above A, passes through an eyelet fixed on the rod at the end B, and carries a mass

of weight w at its other end. Prove that, in the absence of friction, the rod will be in equilibrium when BC is, if possible, equal to $\frac{w.AC}{\frac{1}{2}W+w}$. Prove also that this position of equilibrium is an unstable one.

44. A uniform rod AB, of length a and weight W, is suspended in a horizontal position, by two light strings AD and BE, each of length l, which are parallel and inclined at a given angle α to the vertical, and by another light string AF, which passes to a point F in DE produced, so that $EF=DE$. Find the tensions of these strings; and prove that, if l is chosen so that the tension of AF is as small as possible, then this tension will be $W\sin\alpha$ or $\frac{1}{2}W\sec\alpha$, according as α is $<$ or $>\frac{1}{4}\pi$.

45. A fine rod of length $2a$, with its centre of gravity at a distance c from the centre, is supported at its two ends by a fine light string of length l passing over a smooth pulley of radius a, which is free to turn about a fixed horizontal axis through its centre. Prove that, in the position of equilibrium, the length $2a\gamma$ of the curved portion of the string is given by the equation

$$\{2a(\gamma-\tan\gamma)-l\}^2(a^2+c^2\tan^2\gamma)=4a^4\sec^2\gamma.$$

46. A smooth lamina, in the form of an equilateral triangle ABC, is placed on a horizontal table, so that each of its sides touches one of three smooth pegs P, Q, R, where

$$BP:PC=CQ:QA=AR:RB=n:1.$$

A couple, of moment G, acts on the lamina; show that the pressure on each peg is $\frac{2}{3}\left(\frac{n+1}{n-1}\right)\frac{G}{BC}$. Explain the result when $n=1$.

47. A uniform triangular lamina ABC, right-angled at C, is suspended in a vertical plane with the point B resting against a smooth vertical wall by a fine string of length l attached to C and to a point in the wall situated vertically above B; prove that the angle θ which the string makes with the wall is given by the equation

$$l\tan\theta=(\tan B-3\tan\theta)\{a^2+(a^2-b^2)\tan^2\theta\}^{\frac{1}{2}},$$

where a is the length of the side BC.

48. A uniform rod, of weight W, rests with its lower end A and its upper end B in contact respectively with smooth horizontal and vertical walls, and one point C is attached by a fine string to O, the point of junction of the walls in the vertical plane through the rod. Show that, if the rod rests at an inclination α to the horizontal, the tension of the string is $\frac{1}{2}\dfrac{W\cos\alpha}{\sin(\alpha-\theta)}$, where $\tan\theta=\dfrac{AC}{BC}\tan\alpha$. What limitation is there on the ratio $\dfrac{AC}{BC}$?

49. Three equal circular discs, each of weight W, are surrounded by a light string in a state of tension, and the two lowest rest on a horizontal plane, the centres of the discs forming an equilateral triangle in a vertical plane. If the string is just so tight that the two lower discs just remain in contact, show that its tension is $\frac{1}{2}W\sqrt{3}$.

50. Two circular cylinders, of radii $3a$ and a, are glued together along a generating line, and the compound body thus formed rests with the larger cylinder on a horizontal plane and both cylinders in contact with a third, of radius a, which itself rests on the plane. If all the bodies are smooth and the generating lines are all parallel and horizontal, and the weight of the upper small cylinder is W, prove that the total pressure between the two lower cylinders will be $\frac{15-\sqrt{5}}{40}W$.

51. A step-ladder, whose two legs are hinged together at the top and connected by a light cord at their feet, stands on a smooth horizontal plane. Show that (i) as a man mounts it the tension of the cord increases, and (ii) if the cord stretches, the tension in it increases still more.

52. Two uniform rods AB and AC, each of length l and weight W, are freely hinged together at A, and placed with their ends B and C on a smooth horizontal plane. A body of weight W is fastened to one of the rods at a distance b from its lower end, and equilibrium is maintained by fastening a light string to the rods symmetrically. If the string can only bear a tension equal to W, show that the maximum angle between the rods is $2\tan^{-1}\frac{2l}{l+b}$, and find the length of string that must be available to give this angle.

53. Two uniform rods, each of length $2a$ and weight W, are hinged together at one extremity and placed symmetrically over a smooth circular cylinder fixed with its axis horizontal, the other extremities of the rods being connected by a fine light string also of length $2a$. Show that, if the cylinder is of radius r, then the tension of the string is

$$\frac{W(4r\sqrt{3}-a)}{2a\sqrt{3}}.$$

54. A smooth hemisphere, of radius r, is fixed with its base on a smooth horizontal plane. Two uniform rods, each of length $2a$ and equal in weight, are smoothly hinged together at their upper extremity and rest symmetrically upon the hemisphere. Show that, in the position of equilibrium, the rods will or will not have their ends in contact with the plane according as $2a\sqrt{2}$ is $>$ or $<3r$.

55. A horizontal girder has four bays AB, BC, CD, DE, each 5 feet; it is stiffened by three vertical members BB', CC', DD', each 3 feet, by horizontal members $B'C'$, $C'D'$, and by oblique members AB', $B'C$, $C'D$, $D'E$. Find by graphical construction the tensions and thrusts produced in the members when a uniformly distributed load, of weight W, is supported by the girder.

56. Two rods AB and CD are each capable of motion, in the same horizontal plane, about smooth pivots at their middle points. A, C and B, D are connected by light strings in a state of tension. Show that, when in equilibrium, either the rods or the strings must be parallel.

57. A triangle ABC, formed of three light rods freely jointed together at their extremities, is in equilibrium under the action of

three forces, each perpendicular to the side on which it acts. The lines of action of the forces meet in O, and the point P is taken such that the middle point of OP is the circum-centre of the triangle. Show that the forces and reactions at the joints are all represented in the stress-diagram $PABC$.

58. A smooth circular disc just fits inside a triangle ABC, which is formed of three light rods freely hinged together at their extremities, and the whole is in equilibrium under the action of no external forces; prove that the reactions at A, B, C are proportional to $\sqrt{1-\sin B \sin C}$ and two similar expressions.

Prove also that the direction of the action at A makes with the bisector of the external angle at A, an angle

$$\tan^{-1}\left\{\frac{\sin \frac{1}{2}(B-C)}{\sin \frac{1}{2}A}\right\}.$$

59. Three light rods are freely jointed together at their extremities to form a triangle ABC. An endless stretched elastic string passes through three small rings fixed at the middle points of the sides of the triangle. Prove that the following construction will give the direction of the action at A:—Produce the lines AB and AC beyond B and C to B' and C' respectively, where BB' and CC' are each equal to BC; then $B'C'$ is parallel to the action at A.

60. Three light rods of equal lengths are freely jointed together at their extremities to form a triangle. The middle points of the rods are connected by light stretched strings, the tensions of which are T, T, $2T$. Find the magnitudes of the actions at the joints.

61. Three smooth light rods are freely jointed together at their extremities to form an acute-angled triangle ABC, and on each slides a small smooth ring of no appreciable weight. An endless light elastic string passes through the rings and is stretched with tension T. Show that the actions at the joints are perpendicular to and proportional to AO, BO, CO, where the circum-centre bisects the distance between O and the orthocentre. Prove also that the reaction at A is of magnitude

$$2T\sqrt{\sin^2 A - 4\cos A \cos B \cos C}.$$

62. Three light rods are freely jointed together at their extremities to form a triangle ABC, and the whole rests on a smooth horizontal table. Three light strings in a state of tension, tied to BC, CA, AB at D, E, F respectively, have their other ends tied together at O. Prove the following construction for the direction AL of the action at A:—Draw BH and CK each parallel to OD to meet OF and OE respectively in H and K, and let HK cut BC in L.

63. The extremities of a uniform straight rod, of weight W, are freely hinged to two equal and similar rods, the farther extremities of which are connected by a light string whose length is twice that of a rod. The system is supported symmetrically by two small smooth pegs in a horizontal line at a distance m times the length of a rod, on which the extreme rods rest symmetrically. Show that the tension of the string is $\frac{W\sqrt{3}}{6}(12m-13)$, and find the stress at a hinge.

64. Four equal uniform rods AB, BC, CD, DA, each of weight W, are smoothly jointed together at their extremities. The middle points of AB and BC are constrained by two smooth fixed points in the same vertical, so that $ABCD$ is a square. Prove that the actions at B and D are horizontal and equal to $\frac{3}{2}W$ and $\frac{1}{2}W$ respectively. Find also the magnitudes and directions of the actions at A and C.

65. Four rods, freely jointed together at their extremities to form a quadrilateral, rest on a smooth horizontal table under the action of forces at their middle points perpendicular to them. Prove that a circle can be described about the quadrilateral.

Also, if the forces are proportional to the lengths of the rods on which they act, show that the actions at the joints are all equal and are in the directions of the tangents to the circle at the corresponding points.

66. A rhombus $ABCD$ is formed of four light rods freely jointed together at their extremities. The angular points B and D are connected by a light string, and the system is strained by means of a second light string connecting points in AD and BC, and perpendicular in direction to these sides. If T denotes the tension in the latter string, and α the angle between the sides AB and BC, show that the tension in BD is of magnitude $T\cos\alpha\operatorname{cosec}\frac{1}{2}\alpha$, and determine the thrusts in the rods AB and CD.

67. Five light rods AB, BC, CA, DC, DB in the same plane are smoothly jointed together in the form of two triangles ABC and DBC, on the same base BC, and on the same side of it, AD being parallel to BC. The middle points of AC and BD are joined by a light string at tension T. Show that the stress in the rod BC is of magnitude

$$\frac{AD+BC}{2(BC)}T.$$

68. Five equal uniform rods are freely jointed together at their extremities to form a pentagon $ABCDE$. AB is fixed horizontally, and AD is joined by a light string whose length is equal to that of each rod. Prove that, in the position of equilibrium, the string is inclined to the vertical at an angle $\tan^{-1}(\frac{1}{5}\sqrt{3})$, and that the ratio of its tension to the weight of each rod is $16:5\sqrt{7}$.

69. A see-saw consists of a plank of weight w laid across a fixed rough log whose shape is that of a horizontal circular cylinder. The inclination to the horizontal at which it balances is increased to α when loads of weight W and W' are placed at the lower and higher ends respectively; and the inclination is reduced to β when the loads are interchanged. Show that the inclination of the plank when unloaded is

$$\frac{w'(W+W'+w)(W'\alpha-W\beta)}{w(W+W'-w')(W-W')},$$

where w' is the weight of a load which, placed at the higher end, would balance the plank horizontally.

70. Three forces, acting at a point, are represented in magnitude and direction by AA', BB', CC'. Show that their resultant is represented in magnitude and direction by $3(GG')$, where G and G' are the centroids of the triangles ABC and $A'B'C'$ respectively.

71. Parallel forces P, Q, R, S, T, U act in the same sense at the angular points of a regular hexagon taken one way round. Find the conditions which must hold, in order that the centre of the forces may coincide with the centre of the hexagon.

Art. 133 **72.** Three uniform pieces of wire, of equal weights and unequal lengths, are joined together at their extremities to form a triangle. Find its centre of gravity.

73. A uniform wire is bent into the shape of a triangle ABC. M is the point of concurrence of the medians, and I the centre of the inscribed circle of ABC. Prove that the centre of gravity G of the wire is in IM produced, where $IM = 2(MG)$.

74. Two equal uniform rods AB and BC, of total weight W, are freely jointed together at B, and have their middle points connected by a light inelastic string of such length, that when it is straightened, the angle ABC is a right angle. If the system is suspended by a string attached to the point A, prove that the inclination of AB to the vertical will be $\cot^{-1}3$, and find the tension of the first string and the action at the joint.

75. Three uniform pieces of wire, each of length a and of masses $M+m$, $M-m$, $2M$, are joined together at their extremities to form a triangle, which is suspended by the corner opposite to the heaviest side. Find the depth of the centre of gravity below this corner.

76. Show how to place three particles of given masses on the circumference of a circle, so that its centre may be their centre of mass. What condition must the masses satisfy, in order that this may be possible?

77. A number of small rings can slide freely on a smooth fixed circular wire, and each ring repels every other ring with a force which is measured by the product of their masses and the distance between them; show that, in equilibrium, the centre of mass of the rings is the centre of the circle.

78. A number of equal rectangular blocks, each of length na, are piled one upon another, and each projects a distance a in the direction of its length beyond the one below it. Show that (i) if n is an integer, not more than n blocks can stand, and (ii) if n is not an integer, then the greatest number of blocks that can stand is the integer next above n.

79. A man wishes to support a given load as far over the edge of a cliff as possible by means of a number of planks, of different lengths, whose weights are proportional to their lengths. Show that he must place the planks one upon another in order of their lengths, the longest being uppermost, and determine the maximum distance of the load from the edge, when $a_1, a_2, a_3, \ldots\ldots a_n$ are the lengths of the planks, and b is the length of a plank whose weight is equal to that of the given load.

80. A, B, C, K are four points in one plane such that K is the centroid of three masses placed at A, B, C, proportional respectively to KA, KB, KC. Prove that

$$\frac{BC^2 \cdot KL}{MN} = \frac{CA^2 \cdot KM}{NL} = \frac{AB^2 \cdot KN}{LM},$$

where L, M, N are the feet of the perpendiculars from K upon BC, CA, AB respectively.

81. A triangular lamina, of weight W, is supported by three vertical strings attached to its angular points, so that the plane of the triangle is horizontal. A particle of weight equal to W is placed on the triangle at its orthocentre. Prove that the tensions of the strings are given by

$$\frac{T_1}{1+3\cot B\cot C}=\frac{T_2}{1+3\cot C\cot A}=\frac{T_3}{1+3\cot A\cot B}=\tfrac{1}{3}W.$$

82. A series of cubes, of the same material, are piled one above another on a horizontal plane. The lengths of the sides of the cubes, beginning with the lowest, are $2a$, $2ar$, $2ar^2$, $2ar^3$, and so on in geometrical progression. If there are n cubes in the pile, find the height of the centre of gravity of the whole system above the horizontal plane. If the number of cubes is infinite, r being less than unity, prove that the height of the centre of gravity of the pile is

$$a\left(\frac{1+r^3}{1-r^4}\right).$$

83. A uniform rectangular board $ABCD$, of weight W, is supported in a horizontal position on three pegs placed at the corner B and the middle points of the sides AD and CD. Find the pressures on the pegs, and prove that they are independent of the form of the rectangle.

84. If the centre of gravity of a quadrilateral area is the same as that of four equal particles placed at its angular points, show that it is a parallelogram.

85. If the diagonals of a plane quadrilateral are at right angles, prove that the distance of the centre of gravity of the quadrilateral area from the intersection of the diagonals is two-thirds of the distance between their middle points.

86. A quadrilateral $ABCD$ circumscribes a circle whose centre is O, and the diagonals intersect in E. Show that the centre of gravity of the perimeter of the quadrilateral coincides with the centre of six equal parallel forces acting through A, B, C, D, E, O; the forces through E and O being opposite in sense to the other four.

87. Four light rods are freely jointed together at their extremities to form a quadrilateral. The middle points E and F of one opposite pair are connected by a light string in a state of tension; the middle points G and H of the remaining pair are connected by a light rod in a state of compression. Show that

$$\text{the tension : the thrust} = EF : GH.$$

88. Four equal uniform rods, each of length a, are smoothly jointed together at their extremities to form a square, and opposite joints are connected by light strings, each slightly longer than $a\sqrt{2}$. The two upper rods rest on two smooth pegs which are in a horizontal line, $2c$ apart. Prove that the horizontal or vertical string will be taut according as

$$a \text{ is } < \text{ or } > 2c\sqrt{2}.$$

89. A uniform rod, of length l and weight W, has its extremities

supported by two fine strings, of equal length, attached to two fixed points in the same horizontal line at distance l' apart, where l' is $< l$. A second uniform rod, of length equal to l' and weight W'', has two small smooth rings at its extremities through which the strings pass. Prove that, in equilibrium, the distance between the rods is

$$\tfrac{1}{2}(l-l')\frac{W''}{\sqrt{W''(W''+2W)}}.$$

90. A uniform rod, of length a, is fitted with small smooth rings at each end, and the rings can slide on two smooth straight rods, which are fixed in a vertical plane and inclined at angles α and β to the horizontal. Prove that, in the position of equilibrium, the ring on the rod of inclination α is at a distance x from the joint, where

$$x\sin(\alpha+\beta)\sqrt{\{\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos(\alpha+\beta)\}}$$
$$=a\{\sin\alpha-\sin\beta\cos(\alpha+\beta)\}.$$

91. Three equilateral triangles of jointed rods AOB, COD, EOF, in one plane, are freely jointed together at the point O, and the rhombuses $OBPC$, $ODQE$, $OFRA$ are completed with six other rods freely jointed together at their extremities. Show that, if stresses are induced in the system by a stretched light string connecting P and Q, and a light rod connecting Q and R, then the stress in the rod will be equal to the tension of the string.

92. In a wheel and axle of known radii, if the axle is not concentric with the wheel, show that observation of the greatest and least forces required to support the "weight" would enable us to find not only the "weight" but also the distance between the centres of the axle and the wheel.

93. A number of smooth pulleys, each of weight w, are arranged in the system in which each pulley hangs by a separate string, all the strings being attached to the same fixed beam, and the top string passing over a fixed pulley to allow of the "Power" being applied downwards. Prove that the weights of the pulleys may be counterpoised by attaching a mass of weight w to the lowest pulley and a mass of weight w to the string intended to support the "Power."

94. In a system of n pulleys, in which the same string passes round all the pulleys, the total load supported, including the lower block of pulleys, is of weight W. The end of the string to which the "Power" is usually applied passes under a movable pulley of weight w, over a fixed pulley, and is then attached to the lower pulley-block. If the movable pulley is attached by a string to the lower pulley-block, and if all straight portions of string are vertical, prove that, for equilibrium, $2W$ must be $> (n+1)w$.

95. In a system of n pulleys, all of no appreciable weight, and hanging by separate strings whose fixed ends are parallel and inclined to the vertical at an angle θ, show that the mechanical advantage is $\dfrac{\sin 2^n\theta}{\sin\theta}$.

Show, also, that if the "Power" is applied horizontally, then the mechanical advantage is $\tan\left(\dfrac{2^{n-1}-1}{2^n-1}\pi\right)$.

96. If a portion of given weight is broken off from the end of the long arm of a common steelyard, find the correction which must be made to the apparent weight of a body as indicated by the graduations.

97. A common steelyard is graduated on the assumptions that its weight is Q, and that the movable mass is of weight P, both which assumptions are incorrect. If two masses whose real weights are W_1 and W_2 appear to weigh W_1+w_1 and W_2+w_2 respectively, show that the weight of the movable mass and the weight of the steelyard are less than their assumed values by

$$\frac{P}{R}(w_1-w_2) \text{ and } \frac{Q}{R}(w_1-w_2)+\frac{a}{b}\left(\frac{W_1w_2-W_2w_1}{R}\right),$$

where b and a are the distances from the fulcrum to the centre of gravity of the bar and to the point of attachment of the substance to be weighed, and $R=W_1-W_2+w_1-w_2$.

98. The centre of gravity of a balance is not symmetrically situated with respect to the arms. If the inclination of the beam to the horizontal is θ when masses of equal weight W are placed in the scale-pans, and if it is θ' for masses of equal weight W', prove that the beam will be horizontal when the difference between the weights of the masses in the two scale-pans is

$$\frac{2c(W-W')}{a(\cot\theta-\cot\theta')},$$

where a is the length of each arm, and c is the distance of the fulcrum from the straight line which joins the points of attachment of the scale-pans.

99. A balance has scale-pans of different weight, and arms of different length, but when the scale-pans are empty the beam is horizontal and its centre of gravity is in the same vertical as the fulcrum. When masses of weight P and Q are placed in the scale-pans, the inclination of the beam to the horizontal is θ, and when the masses are interchanged, it is ϕ (on the same side); prove that the lengths of the arms are in the ratio

$$P\tan\theta-Q\tan\phi : Q\tan\theta-P\tan\phi.$$

100. A body can be just sustained upon a rough inclined plane by a force P acting along the plane, or by a force Q acting horizontally; show that the weight of the body is $PQ(Q^2\sec^2\lambda-P^2)^{-\frac{1}{2}}$, where λ is the angle of friction.

101. A uniform square lamina is placed with its plane vertical and one side resting along the line of greatest slope of a rough inclined plane, the coefficient of friction being μ; a string is attached to the uppermost corner and pulled in a direction parallel to the line of greatest slope up the plane. Show that, if the tension is gradually increased, the lamina will slide or tilt according as the angle of inclination of the plane is $<$ or $>\tan^{-1}(1-2\mu)$.

102. A heavy uniform sphere rests on a rough inclined plane, the coefficient of friction being μ, and is supported by a horizontal string attached to the highest point of the sphere and to a point in the plane. Find the inclination of the plane to the horizontal when the sphere is on the point of slipping.

103. Two equal masses, connected by a light string, rest in equilibrium one on each of two equally rough inclined planes of the same height placed back to back, the string passing over a smooth pulley at the top of the planes. Show that the difference of the inclinations of the planes cannot exceed twice the angle of friction.

104. Two heavy particles are connected by a light string passing over a smooth pulley, which is fixed on one horizontal edge of a smooth triangular prism, so that one particle rests on each of the faces which meet in that edge. If the prism becomes rough, show that it may be turned slowly through the angle of friction about a horizontal axis before the particles begin to slip.

105. A reel, consisting of a spindle of radius c with two circular ends of radius a, is placed on a rough inclined plane, and has a thread wound on it which unwinds when the reel rolls downwards. If μ is the coefficient of friction, and α the inclination of the plane to the horizontal, show that the reel can be drawn up the plane by means of the thread provided μ is not $< \dfrac{c \sin \alpha}{a - c \cos \alpha}$.

106. One end of a straight uniform heavy rod rests upon a rough horizontal plane, the other end being connected with a fixed point by a light string. If θ, ϕ, ψ are the inclinations of the string, the rod, and the total resistance of the horizontal plane respectively to the vertical, show that

$$\cot \theta + 2 \cot \phi - \cot \psi = 0.$$

107. Prove that the coefficient of friction between the material of a rough horizontal plane and that of a rod of another material, may be found by placing the rod with one end in contact with the plane, supporting the upper end by a light string of which the other end is fixed, and observing the angles of inclination to the plane of the rod and string when the rod is in limiting equilibrium.

108. A pair of steps of weight W (which may be regarded as two equal uniform beams jointed together at one extremity and connected by a light cord, which is fastened to points on the beams equally distant from their common extremity) is placed on a rough horizontal plane. Find (i) when equilibrium is possible with the cord slack, (ii) the tension in the cord when it is tight.

109. A pair of equal uniform step-ladders stands on a rough horizontal plane with each of its ladders on the point of slipping. Show that the feet of the ladders are at a distance $\dfrac{4\mu l}{\sqrt{1+4\mu^2}}$ apart, where μ is the coefficient of friction and l is the length of each ladder.

110. Two uniform rods AB and AC, of weight W and W' respectively, are freely jointed together at A, and rest in a vertical plane with B and C on a rough horizontal table, their inclinations to the vertical being θ and θ' respectively. Find an expression for the coefficient of friction if the friction at B is limiting.

111. Two uniform rods AB and AC, of weight W and W' respectively, are freely jointed together at A, and rest in a vertical plane with the ends B and C on a rough horizontal plane, their inclinations to the vertical being θ and θ' respectively. Prove that the direction

of the action at the hinge makes an angle $\cot^{-1}\left(\frac{W'\cot\theta - W\cot\theta'}{W+W'}\right)$ with the vertical.

Show also that, if the coefficient of friction is gradually diminished until slipping takes place, then the end B will slip first if $W\cot\theta$ is $< W'\cot\theta'$.

112. Two equal uniform ladders are freely jointed together at one end, and stand with the other ends on a rough horizontal plane. A man, whose weight is equal to that of each ladder, ascends one of them. Prove that the other will slip first.

If it begins to slip when he has ascended a distance x, prove that the coefficient of friction is $\left(\frac{a+x}{2a+x}\right)\tan\alpha$, where a is the length of each ladder and α the angle each makes with the vertical.

113. A uniform plank, of length $2b$, rests with one end on a rough horizontal plane, touches a smooth fixed circular cylinder of radius a lying on the plane, and makes an angle 2α with the plane. Show that, if the plank is at right angles to the axis of the cylinder, and λ is the angle of friction, equilibrium is possible only when

$$a\sin\lambda > b\tan\alpha\cos 2\alpha\sin(2\alpha+\lambda).$$

114. A uniform solid rectangular block leans against a rough vertical wall, with one edge resting on a rough horizontal plane. If θ is the angle which the plane containing the two supporting edges makes with the vertical in the position of limiting equilibrium, and if λ is the angle of friction for both edges, prove that the thickness of the block measured perpendicular to this plane must bear to the distance between the edges the ratio

$$\tan\theta - \tan 2\lambda : \sec 2\lambda.$$

115. Two rough planes, which intersect in a horizontal line, are inclined at angles α and β to the horizontal. A cylindrical ruler, whose length is $2a$ and diameter $2b$, rests with one end on each plane, the axis of the ruler being at right angles to the line of intersection of the planes. Show that, if the ruler is in limiting equilibrium with its axis inclined at an angle θ to the horizontal,

$$\cos(\theta+\phi)\cot(\alpha+\lambda) - \cos(\theta-\phi)\cot(\beta-\lambda) = 2\sin\theta\sin\phi,$$

where $\tan\lambda$ is the coefficient of friction for each end and $\cot\phi = \frac{a}{b}$.

116. Across the top of a fixed horizontal circular cylinder, and at right angles to the axis of the cylinder, rests a uniform plank, of weight W and of length equal to the perimeter of the cross section of the cylinder. Prove that a mass of weight w ($< W$), placed on the plank at the point of contact, can be gradually moved to any position on the plank without causing it to slip, provided the coefficient of friction exceeds

$$\tan\left(\frac{w}{W+w}\pi\right).$$

117. A heavy circular cylinder rests on a rough horizontal plane, and a heavy beam, movable freely about a hinge in that plane, rests at right angles to the axis of the cylinder, the beam touching the rough surface of the cylinder. If the coefficients of friction for the

cylinder and the plane and for the cylinder and the beam are equal, and if the position is one of limiting equilibrium, show that the cylinder will be on the point of rolling along the plane and sliding along the beam.

118. A heavy circular cylinder rests on a rough horizontal plane, and a uniform heavy rod, freely hinged at one end to a fixed point in the plane, rests at right angles to the axis of the cylinder, the middle point of the rod being in contact with the rough surface of the cylinder. If λ is the angle of friction both for the cylinder and the plane, and for the cylinder and the rod, then in limiting equilibrium λ is half the inclination of the rod to the horizontal; and if λ is slightly less than this the cylinder will begin to turn about its lowest point.

119. A rough circular cylinder is fixed with its axis horizontal, and a body rests upon it, having two points in contact with it in a plane perpendicular to the axis, its centre of gravity being in the same plane and on the surface of the cylinder midway between the points of contact. Prove that, if the body is on the point of slipping, the inclination to the vertical of the plane through its centre of gravity and the axis of the cylinder is

$$\cot^{-1}\{(\mu+\frac{1}{\mu})\cos\alpha-\mu\},$$

where μ is the coefficient of friction, and 2α the angular distance between the points of contact.

120. Two equal homogeneous spheres, each of weight W and radius r, rest on a rough horizontal plane. A uniform rod, of weight w, has its extremities freely jointed one to each sphere, and rests in a horizontal position symmetrically between the spheres. Prove that the distance of the rod from the plane must exceed

$$\frac{2rw^2}{\mu^2(2W+w)^2+w^2},$$

where μ is the coefficient of friction between each sphere and the plane.

121. The handles of a drawer are equidistant from the sides of the drawer, and distant c from each other. Prove that it will be impossible to pull the drawer out by pulling one handle unless the length of the drawer exceeds μc, where μ is the coefficient of friction.

122. A sphere, of radius a, is fixed on a smooth horizontal table. A uniform hollow cylinder, of radius c and weight W, stands on the table surrounding the sphere and touching it at one point. Another uniform sphere, of radius b and weight w, is placed inside the cylinder, touching it and the first sphere. Prove that the cylinder cannot remain vertical if all the surfaces are smooth and

$$cW<w(2c-a-b).$$

Show that the cylinder will remain vertical, however light it may be, if there is friction between the fixed sphere and the cylinder and the coefficient is not less than

$$\sqrt{\frac{a+b-c}{c}}.$$

123. Two wedges, in all respects similar, rest on a horizontal table, so that one face of each forms an inclined plane of inclination α to the horizontal. The wedges are placed in contact along the lower edges of these faces, and a uniform circular cylinder rests between them so as to be supported by the faces. The cylinder is of weight W, each wedge of weight w, and the angle of friction between the cylinder and each of the faces is λ. Prove that equilibrium is impossible unless either α is less than λ, or the coefficient of friction between the wedges and the table is greater than

$$\frac{W\tan(\alpha-\lambda)}{W+2w}.$$

124. Two equal and similar isosceles wedges, each of weight W and vertical angle 2α, are placed side by side with their bases on a rough horizontal table, so as to be just in contact along an edge. A uniform smooth sphere, of weight w, is supported between them, being in contact with a face of each. Prove that for equilibrium, it is necessary that the coefficient of friction should be greater than

$$\frac{w\cot\alpha}{2W+w},$$

and that the radius of the sphere should be less than

$$2a\sin\alpha\tan\alpha\left(1+\frac{W}{w}\right),$$

where $2a$ is the length of each base.

125. Two equal uniform cylinders, of radius a, are placed with their axes parallel on a rough horizontal table. A third uniform rough cylinder, of radius b, rests on them with its axis parallel to their axes. The angle of friction between the upper cylinder and each of the others is λ, and the angle of friction between each of the lower cylinders and the table is λ'. Prove that the maximum distance between the axes of the lower cylinders is the smaller of the two quantities

$$2(a+b)\sin 2\lambda \quad \text{and} \quad \frac{4(a+b)wW\tan\lambda'}{w^2+W\tan^2\lambda'},$$

where W is the sum of the weights of the three cylinders, and w is the weight of the upper one.

126. A body of mass M is free to move about a fixed point, from which its centre of gravity is at a distance h. When the centre of gravity is vertically below the point of support, the upper surface of the body, which is plane, is horizontal at a height h' above the centre of gravity. Show that the body can be adjusted to support a rough particle of mass m on its plane surface, if the particle is placed anywhere within a circular area of radius

$$\mu\frac{(M+m)h-mh'}{m},$$

where μ is the coefficient of friction.

127. An odd number of equal heavy particles are knotted at equal intervals on a light string, which rests on the convex arc of a rough vertical circle. Show that, if there is no friction between the string and the circle, the greatest inclination to the horizontal of the line

joining the extreme particles is the angle of friction between any particle and the circle.

128. A uniform circular cylinder, of radius c, rests in equilibrium upon two rough uniform circular cylinders, of radii a and b. The two latter cylinders lie on a rough horizontal table and are just not in contact with each other. Prove that, if a is not less than b, then the coefficient of friction (which is the same for all surfaces in contact) must not be less than

$$\frac{a^2+ac-bc}{\sqrt{ab}\{a+c+\sqrt{(a+b+c)c}\}}.$$

Show also that $c(a-b)$ must be less than b^2.

129. A uniform rod is bent into the form $ABCD$, so that the angles ABC and BCD are right angles, and the parts projecting from BC point in the same direction. Prove that the body will rest with any point of AB resting on a rough peg, if the coefficient of friction exceeds

$$\frac{a^2+c^2}{b^2+2bc}$$

and c is $> \frac{1}{2}\{a+\sqrt{a^2+4ab}\}$,

the lengths of AB, BC, CD being a, b, c respectively.

130. A uniform rod is bent into the form of a triangle ABC, whose sides are of lengths a, b, c, and the whole is hung over a rough peg so that the side BC may rest in contact with it. Find the length of the portion of the rod over which the peg may range, showing that if μ is the coefficient of friction, and p the length of the perpendicular from A upon BC, then the triangle will rest in any position provided μ is greater than

$$\frac{a(a+b+c)}{p(b+c)}+\tan\tfrac{1}{2}(B\sim C).$$

ANSWERS TO THE EXAMPLES

EXAMPLES I

1. AP is vertical.
2. 20, 15, 7 pounds' weight.
3. 11 pounds' weight.
5. 15 feet; 61 pounds' weight.
6. 66 and 114 pounds' weight.
7. 51·4, 39, 21·1 pounds' weight.

EXAMPLES II

1. 8·48 . . . pounds' weight.
2. 5 pounds' weight, acting along OC.
   5 pounds' weight, acting along OD, where D is in BC at a distance of 9 inches from B.
4. (i) 25 pounds' weight; $\frac{7}{24}$.
   (ii) 26 pounds' weight; $\frac{5}{12}$.
   (iii) 30 pounds' weight; $\frac{3}{4}$.
   (iv) 51 pounds' weight; $\frac{8}{15}$.
   (v) 74 pounds' weight; $\frac{12}{35}$.
   (vi) 97·5 pounds' weight; $\frac{16}{63}$.
   (vii) 34·8 pounds' weight; $\frac{20}{21}$.
   (viii) 47·85 . . . pounds' weight; $\frac{21}{43}$.
5. (i) 63 pounds' weight.
   (ii) 39 pounds' weight.
   (iii) 25 pounds' weight.
   (iv) 18·2 pounds' weight.
7. (i) 10 pounds' weight.
   (ii) 7·65 . . . pounds' weight.
   (iii) 5·18 . . . pounds' weight.
8. (i) 7 pounds' weight; $\frac{11}{14}$.
   (ii) 7 pounds' weight; $\frac{13}{14}$.
   (iii) 13 pounds' weight; $\frac{23}{26}$.
   (iv) 13 pounds' weight; $\frac{23}{26}$.
   (v) 13 pounds' weight; $\frac{11}{13}$.
   (vi) 12 pounds' weight nearly; ·936 nearly.
9. 120°.
10. $\frac{13}{36}$.

11. 12 and 5 pounds' weight.
14. 12 pounds' weight.
15. 5 and 10 pounds' weight; ·44.
18. 60°.
19. 15 and 8 pounds' weight.
20. 60°.
22. 85 pounds' weight; $\frac{3}{4}$.

EXAMPLES III

1. 60°.
2. 100, 28, −28, 80, 60 pounds' weight.
   100 and 120 pounds' weight.
3. (i) 5·18 . . . pounds' weight each.
   (ii) 13·07 . . . pounds' weight each.
   (iii) 10 pounds' weight each.
   (iv) 5·77 . . . and 11·55 . . . pounds' weight.
   (v) 8·17 . . . and 11·15 . . . pounds' weight.
4. 3·22 . . . and 10·76 . . . pounds' weight.
   6 and 11·59 . . . pounds' weight.
5. 26 pounds' weight at an angle $\cos^{-1}\frac{3}{26}$ with OA.
   16 and 14 pounds' weight.
6. 32 pounds' weight; 27·71 . . . pounds' weight.
7. $\frac{1}{3}P\sqrt{3}$ at right angles to P, and $\frac{2}{3}P\sqrt{3}$ at an angle of 30° with P.
9. $\sqrt{P^2+Q^2-2PQ\cos\alpha}$, at an angle $\tan^{-1}\dfrac{P\sin\alpha}{P\cos\alpha-Q}$ with the given component.
10. The other component is a force of 13 pounds' weight, at an angle $\cos^{-1}\frac{1}{26}$ with the given component.
11. $-\frac{44}{125}$; 39 pounds' weight.
16. $OB : OA$.
17. (i) 2 inches from A.
    (ii) 3 inches from A in BA produced.
20. $4\frac{1}{8}$ inches; 30 pounds' weight.
21. $4\frac{1}{2}$ inches; 6·8 pounds' weight.
22. 8 inches from A.

EXAMPLES IV p. 32

2. The magnitude of the resultant is represented by $AB+DC$.
   Its direction is parallel to AB.
   Its line of action passes through the intersection of AC and DB.
3. 2 : 1.
10. 60°.
11. $P\sqrt{3}$, parallel to AF, through the intersection of AC and BF.
13. 11 or 24 pounds' weight.
15. 5 and 7·8 pounds' weight.
17. $\frac{3}{4}$ or $\frac{1}{8}$.
21. 3 or 5 pounds' weight.
23. (i) 3 or 5 pounds' weight.
    (ii) 158° or 142°.

26. 31 pounds' weight.
28. 12 and 9 pounds' weight.
30. 11 and 31 pounds' weight.
32. 11 pounds' weight parallel to CF, where F is the middle point of AB.
33. 3 pounds' weight parallel to FC, where F is the middle point of AB.
34. (i) 4 inches from A.
(ii) 5 inches from A.
(iii) 7 inches from A.
(iv) 9 inches from A.
(v) 7 inches from A in BA produced.

EXAMPLES V

1. (i) The forces cannot be arranged so as to produce equilibrium.
(ii) 90°.
(iii) 120°.
(iv) 180°.
(v) 90°.
(vi) 120°.
(vii) 120°.
(viii) 60°.
2. $W \sin\beta \operatorname{cosec}(\alpha+\beta)$ and $W \sin\alpha \operatorname{cosec}(\alpha+\beta)$.
3. 16 and 20 pounds' weight.
4. 17·32 . . . and 10 pounds' weight.
5. 9·66 . . . pounds' weight each.
6. 29·88 . . . and 81·66 . . . pounds' weight.
8. 30 pounds' weight.
10. 6 and 8 pounds' weight.
12. 8 and 6 pounds' weight.
13. 28 and 96 pounds' weight.
14. $\frac{1}{2}\dfrac{l}{\sqrt{l^2-a^2}}W$.
19. $\frac{7}{30}W\sqrt{15}$, $\frac{11}{60}W\sqrt{15}$.
21. 24 pounds.
22. $\sqrt{W^2+P^2-2WP\cos\alpha}$.
23. 13 inches, 28 pounds' weight.
25. 5 pounds' weight perpendicular to the string.
28. 6 pounds' weight.
35. 7 or 8 pounds' weight, according to the direction of the force applied at A.

EXAMPLES VI

2. 15 pounds' weight;
25 pounds' weight, at an angle $\tan^{-1}\frac{3}{4}$ with the vertical.
3. 45·5 pounds' weight.
4. 70 pounds' weight.
5. (i) $\frac{1}{2}W\sqrt{3}$, $\frac{1}{2}W\sqrt{7}$.
(ii) $\frac{1}{2}W$, $\frac{1}{2}W\sqrt{3}$.

6. $\frac{1}{4}W\sqrt{3}$ perpendicular to the rod ; $\frac{1}{4}W\sqrt{7}$.
7. 5 pounds' weight ;
   9·85 pounds' weight at an angle $\tan^{-1} 2{\cdot}25$ with the horizontal.
8. 2·7 feet ; 26 pounds' weight.
9. 8 pounds' weight.
10. $\frac{16}{15}W$.
11. 100 pounds.
12. W ; $W\sqrt{3}$, at an angle of 60° with FA.
13. 17 pounds' weight ;
    25 pounds' weight, at an angle $\cos^{-1}{\cdot}6$ with the horizontal.
14. $\frac{1}{2}W$; $\frac{1}{2}W\sqrt{3}$ along CA.
15. $\frac{1}{2}W$, $\frac{1}{2}W\sqrt{5}$, at an angle $\tan^{-1}\frac{1}{2}$ with the vertical.
16. $\frac{1}{2}W(\sqrt{3}-1)$.
18. $\frac{3}{2}P$ and $\frac{1}{2}P\sqrt{7}$.
20. 3 and $3\frac{1}{2}$ pounds' weight ; $\frac{13}{77}$.
21. 60°, $\frac{65}{56}W$, $\frac{49}{96}W$.
22. $\frac{1}{2}W\sqrt{17}$, at an angle $\tan^{-1}\frac{1}{3}$ with the horizontal ; $\frac{5}{2}W$.
28. $W\tan\alpha$; $\frac{1}{2}W\sec\alpha$.
29. 20 pounds' weight each.
    Tension in AC = 15 pounds' weight = tension in BC.
    Tension in AB = 7 pounds' weight.
30. $\frac{1}{2}W\operatorname{cosec} 2\alpha$.
    $\frac{1}{2}W\operatorname{cosec}\alpha$.

EXAMPLES VII p. 63

1. 173·2 . . ., 200, 173·2 . . . pounds' weight.
   360·6 . . . pounds' weight, at an angle $\tan^{-1} 1{\cdot}039\ldots$ with AB.
2. A force represented by twice AC.
3. 0.
4. 0.
5. 7 pounds' weight, at an angle $\cos^{-1}\frac{13}{14}$ with the middle force.
6. 7·55 . . . pounds' weight, at an angle $\tan^{-1}{\cdot}115\ldots$ with the middle force.
7. 7 pounds' weight, at an angle $\cos^{-1}\frac{11}{14}$ with the first force.
8. $2\frac{1}{2}$ pounds' weight in direction AC.
9. $2\frac{1}{8}$ pounds' weight in direction CB.
11. $P+Q$.
12. 20 pounds' weight along AE.
15. 7 pounds' weight in the direction of the first force.
16. 15 pounds' weight, at an angle $\tan^{-1}\frac{3}{4}$ with AD.
17. $\frac{1}{2}(Q-P\sqrt{3})$.
18. $X=6{\cdot}35\ldots$, $Y=2{\cdot}31\ldots$.
19. $X=3$, $Y=6{\cdot}93\ldots$.
20. 3·73 . . . pounds' weight along DA.
21. 2·64 . . . pounds' weight, at an angle $\tan^{-1}{\cdot}289\ldots$ with CB.
22. $P=5{\cdot}80\ldots$, $Q=6{\cdot}12\ldots$.
23. $X=(P-Q)\sqrt{3}$, $Y=2P-Q$.
24. $\frac{3}{8}Q\sqrt{10}+P$, along OB.
25. 42·77 . . . pounds' weight, at an angle $\tan^{-1} 2{\cdot}824\ldots$ with the first force.

26. 7·89 . . .
27. 13·2 . . . pounds' weight; $\tan^{-1} 1{\cdot}029 \ldots$
28. $X = Q + \frac{1}{2}(P-R)(\sqrt{5}+1)$, $Y = P + \frac{1}{2}(Q-R)(\sqrt{5}+1)$.

EXAMPLES VIII p. 73

8. The resultant is represented by $4(OE)$, where E is the middle point of BD.
10. $W \tan a$, $W \sec a$, $W \sec a$ in direction AB.
$P \cot a$.
In either of the two directions inclined at an angle $\sin^{-1} \frac{P \cos a}{Q}$ with the rod.
11. $2W$, $\frac{3}{2}W$.
$\frac{1}{6}P\sqrt{10}$ along BD, where D is 8 inches vertically below A.
$\frac{1}{3}P$.
12. $\frac{1}{3}W\sqrt{3}$.
13. $\frac{a}{\sqrt{4c^2-a^2}}W$.
15. $\frac{1}{15}W(10+3\sqrt{5})$.
16. $\frac{7}{24}\frac{7}{6}W\sqrt{15}$.
20. In each case—
In BC, a thrust $W \cot a$.
In CA, a tension $W \operatorname{cosec} a$.
The other forces are—
(i) In AB, a thrust W.
The applied force is $W \cot a$.
The reaction at B is $W \operatorname{cosec} a$ at an angle a with the horizontal.
(ii) In AB, a thrust $W \sec a$.
The applied force is $2W \operatorname{cosec} 2a$.
The reaction at B is $W\sqrt{1+4\operatorname{cosec}^2 2a}$, at an angle $\tan^{-1}(\frac{1}{2}\sin 2a)$ with the horizontal.
(iii) In AB, a thrust $W \operatorname{cosec} a$.
The applied force is $2W \cot a$.
The reaction at B is $W\sqrt{1+4\cot^2 a}$, at an angle $\tan^{-1}(\frac{1}{2}\tan a)$ with the horizontal.
(iv) In AB, a thrust $W \operatorname{cosec} 2a$.
The applied force is $W \operatorname{cosec} 2a$.
The reaction at B is $W\sqrt{1+\operatorname{cosec}^2 2a}$, at an angle $\tan^{-1}(\sin 2a)$ with the horizontal.
(v) The force applied at A is $\frac{16}{15}W\sqrt{3}$.
In BC, a thrust $\frac{1}{3}W\sqrt{3}$.
In CA, a tension $\frac{2}{3}W\sqrt{3}$.
In AB, a thrust $\frac{14}{15}W\sqrt{3}$.
The reaction at B is $\frac{1}{15}\frac{1}{6}W\sqrt{993}$, at an angle $\tan^{-1}\frac{5\sqrt{3}}{16}$ with the horizontal.

21. $W \tan \beta$, $W \tan \gamma$.

22. $2P$.

23. $\frac{c}{\sqrt{c^2-a^2}}W$, $\frac{a}{\sqrt{c^2-a^2}}W$, $\frac{c}{\sqrt{c^2-a^2}}W$.

24. At A, $\frac{Wc \sim W'b}{a}$ in a horizontal direction.

At O, $\sqrt{(W+W')^2+\left(\frac{Wc \sim W'b}{a}\right)^2}$ at an angle $\tan^{-1}\frac{Wc \sim W'b}{a(W+W')}$ with the vertical.

25. Each is of magnitude W. The reaction at O is W vertically downwards.

27. (i) $W' = W = T$.

(ii) $W' = \frac{b}{c}W = T$.

(iii) $W' = W$, $T = \left(\frac{2}{1+\tan\alpha\cot\beta}\right)W$.

(iv) $W' = W \tan\alpha\cot\beta$, $T = \frac{1}{2}(1+\tan\alpha\cot\beta)W$.

28. $\frac{1}{3}W$.
In BA, BD, BC, tensions $\frac{3}{10}W$, $\frac{1}{2}W$, $\frac{2}{5}W$ respectively.
In DA and DC, thrusts $\frac{2}{5}W$ and $\frac{3}{10}W$ respectively.

29. The forces applied at B and C are each $\frac{1}{2}W$ cosec α.
The tension of OA is $\frac{1}{2}W(1-\cot^2\alpha)$.

31. (i) $\frac{1}{10}a$.
(ii) $\frac{2}{3}W$.
(iii) $\frac{7}{24}a$.
(iv) $\tan^{-1}\frac{5}{8}$.
(v) $\frac{1}{12}W\sqrt{73}$.

EXAMPLES IX p. 82

1. $\frac{5}{4}W$, $\frac{3}{4}W$.

2. $W \tan\alpha$, $W \sec\alpha$.

3. $\frac{a}{r}W$, W.

4. $W\cos\alpha$, $W\sin\alpha$.

5. $W\cos\alpha\sec\frac{\alpha}{2}$, $W\tan\frac{\alpha}{2}$.

6. $\frac{2r^2-a^2}{r\sqrt{4r^2-a^2}}W$, $\frac{a}{\sqrt{4r^2-a^2}}W$.

7. $\frac{1}{2}W\sqrt{3}$, at an angle of 30° with the vertical.

8. (i) $W\tan\alpha$, (ii) $W\sin\alpha$; vertically upwards.

11. (i) $216\frac{2}{3}$ pounds' weight, (ii) 200 pounds' weight.

12. W each.

14. 10 pounds' weight; 31·62 . . . pounds' weight.

15. 6 feet $8\frac{51}{74}$ inches.

16. W.

18. $\frac{h'}{h}P$.

20. $W\left(\frac{l\sin\alpha}{\sqrt{l^2-h^2}}\right)$.

22. $W \sin \alpha$.
25. 4 : 5 ; 20 pounds' weight.
26. $\frac{1}{2} W \sec \alpha$ each, at an angle α with the vertical.
27. $\frac{1}{2} W$, $\frac{1}{2} W \sqrt{3}$.
28. $11\frac{1}{3}$ pounds' weight.
29. $60°$, W.
30. $\frac{13}{20} W$, $\frac{21}{20} W$.
31. $\frac{9}{25} d$ or $\frac{16}{25} d$, or $\frac{5}{8} W$ or $\frac{39}{40} W$.

EXAMPLES X p. 89

1. $P\sqrt{2}$, acting at an angle of $45°$ with the vertical.
2. 20 pounds' weight.
3. 20 pounds' weight.
4. 6 and 8 pounds' weight.
5. $\frac{6}{13} W \sqrt{13}$.
6. $W\sqrt{\frac{2(l-h)}{l}}$ and $W\sqrt{\frac{2(l+h)}{l}}$.
7. W at each peg, in a direction making $30°$ with the horizontal.
8. $\frac{1}{2} W \frac{l}{\sqrt{l^2 - c^2}}$.
9. 3·38 . . . ounces' weight.
11. 2·19 . . . ounces' weight ; 3·23 . . . ounces' weight.
12. $\frac{1}{25} W \sqrt{15}$; $\frac{16}{75} W \sqrt{15}$.
13. $\frac{4}{5} W$, $\frac{3}{5} W$.
14. $\frac{b}{\sqrt{2a^2 + 2b^2 - c^2}} W$, $\frac{a}{\sqrt{2a^2 + 2b^2 - c^2}} W$.
15. At C, $W\frac{c}{a}$ parallel to AB.

    At B, $W\sqrt{\frac{2a+c}{a}}$ along the bisector of the angle ABC.
18. $\frac{1}{2} W \sec \alpha$.
    At B and C, $W \tan \alpha$ in a horizontal direction.
    At A, W in a vertical direction.
19. $\frac{5}{6} W$.
    At A, $\frac{1}{3} W \sqrt{5}$.
    At B, $\frac{2}{3} W \sqrt{5}$.
    At C, $\frac{4}{3} W$ in a horizontal direction.
20. Along OC, where O divides AB in the ratio $b : a$.
    At A, $P\sqrt{\frac{a(b+c-a)}{c(a+b-c)}}$.
    At B, $P\sqrt{\frac{b(c+a-b)}{c(a+b-c)}}$.
21. $\frac{w}{2\sqrt{4W^2 - w^2}} a$, $\sqrt{(2W + w)W}$.
22. 2·8 pounds ; 8 pounds' weight.
24. $a \cos \alpha \pm \frac{a}{w} \sqrt{W^2 \tan^2 \alpha - w^2 \sin^2 \alpha}$.

EXAMPLES XI p 95

1. 12 pounds' weight.
2. 40 pounds' weight.
3. $W\cos\frac{\alpha}{2}+w\cos\alpha$, $W\sin\frac{\alpha}{2}+w\sin\alpha$.
4. $\pi-2\alpha$, $\frac{1}{2}W\operatorname{cosec}\alpha$, $l\cot\alpha$.
5. $W\frac{a}{\sqrt{4r^2-a^2}}+w\cdot\frac{a}{r}$.
6. $\frac{War-w(2r^2-a^2)}{r\sqrt{4r^2-a^2}}$.
7. $\frac{1}{3}W\sqrt{3}$; $\frac{1}{3}W\sqrt{3}$, in a direction perpendicular to the rod.
8. $b\sqrt{\sin^2\alpha+\left(\frac{b}{a-b}\right)^2\cos^2\alpha}$; $W\sqrt{1+\left(\frac{a-b}{b}\right)^2\tan^2\alpha}$.
9. Pressure at $A=\frac{a}{h}W$ in a horizontal direction.

   Reaction of the hinge $=\frac{\sqrt{a^2+h^2}}{h}W$ at an angle $\tan^{-1}\frac{a}{h}$ with the vertical.
10. $\frac{1}{2}W$; $\frac{1}{3}W\sqrt{3}$.
11. The middle point; $\frac{2}{3}W$, $\frac{5}{6}W$.
12. K divides BC in the ratio 3 : 5; $\frac{1}{3}W\sqrt{3}$, $\frac{2}{3}W\sqrt{3}$.
13. $\frac{a}{\sqrt{16a^2-b^2}}W$.
14. Perpendicular to BC, 9 inches from B.
15. In direction OP, where P is at distances of 6 and 8 inches respectively from B and C.
16. $\frac{al}{b\sqrt{b^2-l^2}}W$.
17. 4·8 feet; 56 pounds' weight.
20. 120°; W.
25. $\frac{2W^2}{\sqrt{4W^2-P^2}}$.
26. The rod is inclined at an angle $2\sin^{-1}\frac{P}{W}$ with AC.

    The action at the hinge is $\sqrt{W^2-P^2}$ at an angle $\sin^{-1}\frac{P}{W}$ with AC.

EXAMPLES XII 104–106

2. (i) 15 pounds' weight, in the direction of the 9 pounds' weight, along a line which meets AB 12 inches from A.
   (ii) 3 pounds' weight, in the direction of the 9 pounds' weight, along a line which meets BA produced 60 inches from A.
3. (i) 9 pounds' weight, in the direction of the 7 pounds' weight, along a line which meets AB 35 inches from A.
   (ii) 5 pounds' weight, in the direction of the 7 pounds' weight, along a line which meets AB produced 63 inches from A.

4. (i) $12P$, in the direction of the $7P$, along a line which meets AB 7 inches from A.
   (ii) $2P$, in the direction of the $7P$, along a line which meets AB produced 3 feet 6 inches from A.
5. 5 pounds' weight, in the direction of the second force, at a distance of 16 inches beyond that force.
6. (i) $R=8$, $c=5\frac{1}{4}$.
   (ii) $Q=3$, $c=4$.
   (iii) $R=17$, $b=3$.
   (iv) $Q=6$, $b=1$.
   (v) $P=\frac{3}{5}Q$, $R=\frac{4}{5}Q$.
   (vi) $P=\frac{3}{8}R$, $Q=\frac{5}{8}R$.
7. (i) $R=1$, $c=34$.
   (ii) $P=8$, $c=16$.
   (iii) $R=5$, $b=1\frac{1}{2}$.
   (iv) $Q=9$, $b=3$.
   (v) $Q=\frac{2}{3}P$, $R=\frac{1}{3}P$.
   (vi) $P=\frac{5}{3}R$, $Q=\frac{2}{3}R$.
8. 24.
11. 1·895 pounds' weight.
12. Two-thirds and one-third.
13. 96·43 . . . and 53·57 . . . pounds' weight.
14. 100 and 40 pounds.
    The first man would support 12 pounds more, the other 12 pounds less.
15. 6 ounces. No.
17. At a distance $\frac{W-P+Q}{2W}c$ from A.
18. 17·22 . . . and 38·78 . . . pounds' weight.

EXAMPLES XIII 120-124

1. 6 pounds' weight ; infinity.
2. $2\frac{5}{12}$ pounds' weight, acting at an angle $\tan^{-1} 1{\cdot}05\ldots$ with CA.
5. The moments of the two forces about C are in the ratio $\sqrt{45} : \sqrt{44}$, so that rotation takes place about C in the direction BOA.
6. The moments of the two forces about A are in the ratio $\sqrt{128} : \sqrt{125}$, so that rotation takes place about A in the direction DCB.
7. The moments of the two forces about A are in the ratio 100 : 99, so that rotation takes place about A in the direction FDB.
9. (i) $4\frac{1}{2}$ inches.
   (ii) 10 inches.
   (iii) 1 foot.
   (iv) 2 feet.
   (v) 35 pounds' weight.
   (vi) 35 and 28 pounds.
12. $\frac{a-c}{c}W$ and $\frac{a}{c}W$.

14. P is 3 feet from A; Q is 2 feet from B.
15. $6\frac{3}{4}$ feet from A.
16. 20 inches, 12 pounds' weight.
17. 7 pounds' weight, 9·14 . . . inches from A.
18. $\frac{Wa+W_1a_1}{b}$, $\frac{W(a+b)+W_1a_1}{Wa+W_1a_1}a$ from A.
19. $\frac{Pa+Qb}{c-a-b}$, $\frac{(c-b)P+Qb}{Pa+Qb}a$ from A.
20. (i) $\frac{h^2+4r^2}{h^2+r^2}\cdot\frac{W}{4}$.
    (ii) $\frac{1}{4}W$.
22. $\pi-\cos^{-1}\sqrt{\frac{Q}{Q'}}$.
23. (i) 61·09 . . . pounds' weight.
    (ii) 77·59 . . . pounds' weight.
    (iii) 48 pounds' weight.
24. 16 pounds' weight; 40 pounds' weight.
26. 9·49 . . . pounds' weight.
27. $\frac{a\sqrt{2(h^2+k^2)}}{h^2-k^2}W$.
29. $\frac{1}{4}\frac{a}{c}\sec\alpha(1-\tan\alpha)W$.

EXAMPLES XIV

1. $\frac{1}{4}(3W+2w)\tan\alpha$, $W+w$, $\frac{1}{4}(3W+2w)\tan\alpha$.
2. (i) $\frac{3}{8}W+\frac{3}{16}w$, (ii) $\frac{3}{4}W+\frac{5}{16}w$.
4. 17·5 pounds' weight; 22·6 . . . pounds' weight, at an angle $\tan^{-1}\cdot525$ with the vertical.
5. $\frac{9}{5}W$; $3W$, at an angle $\tan^{-1}\frac{3}{4}$ with the vertical.
6. $W\frac{a}{c}\sin\alpha\cos\alpha$.
7. $W\frac{alh}{(l^2+h^2)^{\frac{3}{2}}}$, $W\frac{al}{l^2+h^2}$.
8. $\frac{1}{2}W\frac{a}{b}$, $W\frac{3b^2+c^2-a^2}{4b^2}$.
9. $-\frac{1}{2}P$, $\frac{1}{2}P\sqrt{2}$, $\frac{1}{2}P$.
10. $P=\frac{1}{2}W$, $Q=\frac{1}{2}W\sqrt{2}$, $R=\frac{1}{2}W\sqrt{5}$.
11. $\frac{1}{2}W$, $W(\tan\alpha-\frac{1}{2})$, $W\sec\alpha$.
12. $\frac{7}{10}W$, $\frac{9}{5}W$, $\frac{5}{4}\frac{7}{6}W$.
13. $\frac{c}{d}P$, $\frac{b}{d}P$, $\frac{a}{d}P$.
14. $\frac{1}{4}P$, $\frac{1}{4}P\sqrt{3}$, $\frac{1}{2}P\sqrt{3}$.
15. $\frac{3a-b}{3b}W\sqrt{3}$, $\frac{3a+2b}{6b}W\sqrt{3}$, $\frac{3a}{2b}W$
16. W, $2W\sqrt{3}$, $2W$.
17. $\frac{2a+b}{2a_1+b}W$, $\frac{a+a_1+b}{2a_1+b}W\operatorname{cosec}\alpha$.

18. $\frac{1}{2}W\sin 2\alpha$; $\frac{1}{2}W$, $W(\frac{1}{2}+\cos^2\alpha)$.

19. $\tan^{-1}\left\{\dfrac{b\cot\beta - a\cot\alpha}{a+b}\right\}$; $W\sin\beta\operatorname{cosec}(\alpha+\beta)$, $W\sin\alpha\operatorname{cosec}(\alpha+\beta)$.

20. $\dfrac{W}{2}\left(\dfrac{a\cos\theta - b\sin\theta}{a\sin\theta - b\cos\theta}\right)$, W, $\dfrac{W}{2}\left(\dfrac{a\cos\theta - b\sin\theta}{a\sin\theta - b\cos\theta}\right)$.

21. It divides AB in the ratio $2:3$.

EXAMPLES XV 140 – 143

1. The resultant is represented in magnitude, direction and position by $2AE$, where E is the middle point of BC.

2. The resultant is represented in magnitude, direction, and position by $2BC$.

3. The resultant is represented in magnitude and direction by $2AC$, and acts through the middle point of BD.

4. The resultant is represented in magnitude, direction and position by $4EF$, where E and F are the middle points of AC and BD respectively.

9. $Q\sqrt{3}$, perpendicular to BC, through a point dividing BC externally in the ratio $P+Q:P-Q$.

10. 7 pounds' weight, at an angle $\tan^{-1}\cdot787\ldots$ with BC, through a point D in BC produced, where $CD=BC$.

11. 8 pounds' weight, in direction parallel to BA, through a point situated in BC produced at a distance of 2 inches from C.

12. $3\cdot6\ldots$ pounds' weight along FG, where G is in BC produced such that $CG=\frac{5}{3}BC$, and F is in AD produced such that $DF=AD$.

13. $\sqrt{P^2+PQ+Q^2}$, at an angle $\tan^{-1}\dfrac{P+Q}{P-Q}\sqrt{3}$ with FC, through a point O in FC produced, where $CO=$side of hexagon.

14. $13\cdot2$ pounds' weight along AB.

18. $\theta=0$.

20. $2\sin^{-1}\dfrac{w}{W}$; $\sqrt{W^2-w^2}$ at an angle $\sin^{-1}\dfrac{w}{W}$ with the vertical.

21. $21\cdot79\ldots$ pounds' weight, at an angle $\tan^{-1}\cdot894\ldots$ with the vertical.

23. (i) $W\dfrac{2ab(4a^2-b^2)}{(4a^2+b^2)^2}$, (ii) $W\dfrac{16a^4+b^4}{(4a^2+b^2)^2}$, (iii) $W\dfrac{2ab}{4a^2+b^2}$.

30. (i) $-\dfrac{c}{c_1}P$, $\dfrac{a}{c_1}P$, $-\dfrac{b}{c_1}P$.

(ii) $-\dfrac{c}{c_1}P$, $-\dfrac{a}{c_1}P$, $\dfrac{b}{c_1}P$.

(iii) and (iv) $-\dfrac{c_1}{c}P$, $-\dfrac{a}{c}P$, $\dfrac{b}{c}P$.

(v) $-\dfrac{c}{c_1}P$, $-\dfrac{b}{c_1}P$, $\dfrac{a}{c_1}P$.

(vi) $-\dfrac{c}{c_1}P$, $\dfrac{b}{c_1}P$, $-\dfrac{a}{c_1}P$.

30. (vii) $\frac{c}{c_1}P$, $\frac{b}{c_1}P$, $\frac{a}{c_1}P$.

(viii) $\frac{3c_1}{c}P$, $\frac{a}{c}P$, $\frac{3b}{c}P$.

EXAMPLES XVI 151 – 160

1. $\frac{c}{\sqrt{a^2-c^2}}P$.

2. $\frac{1}{2}W\frac{l}{\sqrt{a^2-l^2}}$, $\frac{1}{2}W\frac{a}{\sqrt{a^2-l^2}}$.

3. At B, a pressure $\frac{c^2}{b^2+c^2}W$.

   At C, a pressure $\frac{b^2}{b+c^2}W$.

   In BC, a tension $\frac{bc}{b^2+c^2}W$.

   In CA, a thrust $\frac{b}{\sqrt{b^2+c^2}}W$.

   In AB, a thrust $\frac{c}{\sqrt{b^2+c^2}}W$.

10. At D, a force W.

    In BC, a thrust $W\left(\frac{a}{b}-1\right)$.

    In AD, a tension $W\left(\frac{a}{b}-1\right)$.

    In DC, a tension W.

11. $W\sqrt{\left(3-\frac{a}{b}\right)}$.

12. $\tan\alpha : \tan\alpha_1$.

18. In OA, a tension W.

    In OB, a tension $\frac{b}{a}W$.

    In OC, a tension $\frac{c}{a}W$

    In AB, a thrust $\frac{c}{a}W$.

    In AC, a thrust $\frac{b}{a}W$.

19. $\left(1-\frac{b^2-c^2}{a^2}\right)W$.

21. In AC, a tension $(W_2 - W_1)\cos\alpha$.
    In CD, a tension $(W_1\cos^2\alpha + W_2\sin^2\alpha)\operatorname{cosec}\alpha$.

22. At A, a pressure $W_1\sin^2\alpha + W_2\cos^2\alpha$.
    At D, a pressure $W_1\cos^2\alpha + W_2\sin^2\alpha$.
    In AC, a tension $(W_1 - W_2)\cos\alpha$.
    In AD, a tension $(W_1\cos^2\alpha + W_2\sin^2\alpha)\cot\alpha$.

23. At B and C, forces $\frac{1}{2}W$ each.

In OA, OB, OC, tensions $\frac{1}{2}W$, $\frac{1}{2}W\frac{b}{a}$, $\frac{1}{2}W\frac{c}{a}$ respectively.

In AB and AC, thrusts $\frac{1}{2}W\frac{c}{a}$ and $\frac{1}{2}W\frac{b}{a}$ respectively.

24. At B and C, forces $\frac{1}{3}W$ and $\frac{2}{3}W$ respectively.

In OA, OB, OC, tensions $\frac{1}{3}W\frac{a}{d}$, $\frac{1}{6}W\frac{b}{d}$, $\frac{2}{3}W\frac{c}{d}$ respectively;

In AB and AC, thrusts $\frac{1}{3}W\frac{c}{d}$ and $\frac{1}{3}W\frac{b}{d}$ respectively; where $2d=\sqrt{2a^2+2c^2-b^2}$.

27. $W\sin\alpha$.

EXAMPLES XVII

1. $10P$ in the direction of the $7P$ and $4P$ and midway between them.
2. 4 pounds' weight in the direction of the last two forces, through D in AC produced, where $CD=3$ inches.
3. $Q=2w$; D is the middle point of AC.
4. (i) 10 pounds' weight, (ii) 15 pounds' weight.
5. 114 and 112 pounds' weight.
6. 33·79 . . . and 33·21 . . . pounds' weight.
7. B is $1\frac{1}{2}$ inches from one peg.
8. 1 pound.
9. $\frac{1}{6}w$.
10. 63 pounds.
11. 5 feet.
12. 3 inches.
13. 42 and 36 pounds' weight.
14. $3\frac{2}{11}$ feet from A.
15. Between $\frac{2}{11}$ and $6\frac{2}{11}$ feet from A.
18. 2 pounds, $6\frac{1}{2}$ inches from A.
19. $\frac{a-b}{2b}.w$.

EXAMPLES XVIII 172 - 174.

1. It divides the distance between the extreme particles in the ratio 7 : 5.
2. $7\frac{1}{3}$ inches from the particle of 11 pounds.
3. It divides the rod in the ratio 13 : 11.
4. 11 inches from A.
5. 2 pounds.
6. $19\frac{1}{5}$ inches from that end.
7. 2 pounds' weight.
9. It divides the straight line joining A to the middle point of BC in the ratio 3 : 2.
10. The mass of the particle at A is three times that of each of the others.
14. Half a side of the triangle.

15. Half a side of the triangle.
16. $\frac{3}{4\sqrt{3}}$ of a side of the square.
17. $\frac{5}{2\sqrt{4}}$ of a side of the square.
18. $\frac{\sqrt{(m_1a+m_2h)^2+(m_2k)^2}}{m+m_1+m_2}$.
20. At the intersection of AD with CE.
22. 15, 10, 7 inches.
23. $\frac{1}{2}a\tan\frac{a}{2}$.
24. $\frac{1}{2}M\cos\alpha\,.\,\sec^2\frac{a}{2}$.
25. 5 and $1\frac{1}{2}$ inches.

EXAMPLES XIX

3. 4·308 . . ., 4, 3·733 . . . inches.
4. $\frac{1}{3}W$, $\frac{2}{3}W$.
5. $\frac{1}{3}M$.
7. $\frac{2}{3}$ inch.
8. ·694 . . . inches on the other side of the centre of the plate.
9. $1\frac{1}{2}$ inches.
10. The centre of the hole must be $7\frac{1}{2}$ inches from the centre of the lamina.
11. 5·72 . . . inches.
12. 25 and 5 inches from the ends of the rod.
13. It divides OE in the ratio 2 : 7, where E is the middle point of CD.
14. It divides HA in the ratio 25 : 38, where H is the middle point of EF.
15. $\sqrt{13} : 2$.
16. It divides the straight line joining one vertex to the middle point of the opposite side in the ratio 7 : 3.
17. 7 inches.
18. 7·9 inches.
21. $1\frac{1}{2}$ inches from A in AB.
22. 1 : 9.
23. $2 : 3n(n+1)$.
24. $\frac{3a^2-h^2}{3(2a+h)}$; $a\sqrt{3}$.
25. $\frac{3a^2-h^2}{3(2a-h)}$; $\frac{1}{2}a(3-\sqrt{3})$.
26. It divides the straight line joining the middle point of the fourth side to the middle point of the opposite side in the ratio 4 : 5.
27. It divides CH in the ratio 74 : 41, where H is the middle point of EF.
28. It divides AD in the ratio 65 : 34, where D is the middle point of BC.

29. It divides AD in the ratio $2(n^3-3n+1):n^3-2$, where D is the middle point of BC.

30. $\frac{1}{2}a(\sqrt{5}-1)$.

EXAMPLES XX 190

1. From a point of trisection of one side.

5. $\frac{1}{2}c(3-\sqrt{3})$.

6. At the intersection of EF with HK, where F is in CD 1 inch from C, H is in AB $1\frac{3}{4}$ inches from B, K is in DA $\frac{2}{3}$ inch from D.

7. $\tan^{-1}\left(\frac{m+m'}{m-m'}\cot\alpha\right)$.

8. $60°$.

10. $\tan^{-1}\left\{\cot\alpha-\frac{b^2}{a(a+2b)\sin\alpha}\right\}$.

13. 6 pounds.

14. 5 pounds.

15. 2 pounds.

16. 7 pounds' weight.

17. $\frac{b^2+3c^2-a^2}{6c^2}W$.

18. 4 pounds.

23. At the middle point of CD, where D divides AB in the ratio $2:1$.

25. $1:\sqrt{3}+1$.

26. 16.

27. $\frac{2}{2}\frac{6}{1}W$.

28. $\left(\frac{a-b}{a+b}\right)\frac{h}{6}$.

29. $\frac{4}{9}W$, $\frac{1}{6}W$, $\frac{7}{18}W$.

34. 20.

35. 6.

EXAMPLES XXI 205-206.

1. $\frac{b^2}{4a}W$.

2. $\frac{b^2}{2a}W$.

3. $\frac{n+1}{2n}W(a-b)$.

4. $\frac{r(r-1)}{2n^2}W(a-b)$.

7. Half the total weight of the rods.

EXAMPLES XXII

1. 10 pounds' weight.
2. 4 inches.
4. 25 pounds' weight.
5. The line joining the two masses must be vertical. $2w\frac{a}{b}\cos\frac{\alpha}{2}$.
6. $\frac{W+w}{2}\left(\frac{a-b}{c}\right)$.
8. Between 0 and $\frac{2}{3}\pi$.
9. $54\frac{1}{3}$ pounds' weight.
10. 42 pounds' weight.
11. 21 pounds' weight.
12. 10 pounds' weight.
13. $\frac{1}{16}(W+15w)$.
14. $\frac{1}{17}(W+15w)$.
15. $2^nP = W + w_1 + 2w_2 + 2^2w_3 + \ldots + 2^{n-1}w_n$.
16. $2^nP = W + w(2^n - 1)$.
17. $\frac{2^n-1}{2^n}(W-w)$.
20. 10 pounds. The upper block contains 3 pulleys and the lower 2 pulleys.
21. 43 pounds' weight.
22. 129·3 pounds' weight.
23. 40 feet.
24. 35 feet.
25. 36 pounds' weight ; 63 inches.
26. 6 feet.
27. $W = 15P + 11w$; $\frac{2}{15}\left\{32 + 23\left(\frac{w}{W}\right)\right\}a$.
28. $W = (2^n-1)P + (2^{n-1}-1)w_1 + (2^{n-2}-1)w_2 + \ldots + (2^2-1)w_{n-2} + (2-1)w_{n-1}$.
29. $W = (2^n-1)P + (2^n - n - 1)w$.
30. $\frac{2^n}{2^n-1}na$.
31. $W + w_1 = 4P + w_2$; 20 inches.
32. $W = P(1+\sqrt{3})$.
33. $4P = W$.
34. The part of the string which is not vertical is inclined to the plane at an angle $\cos^{-1}\frac{W\sin\alpha}{P}$;

    Pressure $= W\cos\alpha - \sqrt{P^2 - W^2\sin^2\alpha}$.
35. 45°.
37. About 10,000 pounds' weight.
38. 45°.
39. 7·92 . . . pounds' weight.

EXAMPLES XXIII

3. (i) 25 pounds. (ii) 20 inches from the fulcrum. (iii) $1\frac{1}{3}$ inches.
4. 7 inches from the end to which the scalepan is attached ; $2\frac{1}{3}$ inches.
5. $\frac{1}{2}$ inch ; 33 pounds' weight.
6. (i) 4 inches from the fulcrum. (ii) $\frac{8}{9}$ inch. (iii) $3\frac{5}{9}$ inches from the fulcrum.
7. (i) $\frac{1}{3}$ inch from the fulcrum in the shorter arm. (ii) 4 pounds. (iii) $\frac{1}{2}$ inch.
8. $x+\dfrac{W'-W}{P}\,a.$
11. $w=\dfrac{2W_0a}{b-a}$; $P=(W_1-W_0)\dfrac{a}{b}$.
15. 15 inches.
16. (i) 3 pounds. (ii) 24 inches. (iii) 9 inches. (iv) 9 pounds.
18. The 15th graduation denotes 1 pound.
    ,, 20th ,, ,, 2 pounds.
    ,, 24th ,, ,, 4 ,,
    ,, 25th ,, ,, 5 ,,
    ,, 27th ,, ,, 9 ,,
    ,, 28th ,, ,, 14 ,,
    ,, 29th ,, ,, 29 ,,
20. (i) 4 pounds' weight. (ii) 8 pounds' weight.
22. $\frac{1}{2}(W_1+W_2)$; $\frac{1}{2}(W_1\sim W_2)$.
23. $W-w$.
24. The arms are of unequal length, one being 1·016 . . . times as long as the other ; 36·895 . . . pounds' weight.
25. 1s. 4d.

EXAMPLES XXIV

1. (i) $\frac{1}{3}W$.

   (ii) $\dfrac{3\sqrt{3}-1}{13}W$.

   (iii) $\dfrac{\sqrt{13}}{4}W$ at an angle $\tan^{-1}\dfrac{\sqrt{3}}{7}$ with the vertical.

   (iv) $\frac{1}{10}W\sqrt{10}$ in an upward direction inclined at an angle $\tan^{-1}\frac{1}{3}$ with the horizontal.

2. (i) $\dfrac{6+5\sqrt{3}}{13}W$.

   (ii) $\dfrac{5\sqrt{3}-6}{13}W$.

   (iii) $\dfrac{3\sqrt{10}+\sqrt{30}}{20}W$, at an angle $\tan^{-1}\frac{1}{3}$ upwards from the plane.

   (iv) $\dfrac{3\sqrt{10}-\sqrt{30}}{20}W$, at an angle $\tan^{-1}\frac{1}{3}$ downwards from the plane.

3. (i) $\frac{2\sqrt{6}-\sqrt{2}}{6}W$.

(ii) $\frac{2\sqrt{2}-\sqrt{6}}{6}W$.

(iii) $\frac{2\sqrt{15}-\sqrt{5}}{10}W$, at an angle $\tan^{-1}\frac{1}{3}$ with the plane.

(iv) $\frac{2\sqrt{5}-\sqrt{15}}{10}W$, at an angle $\tan^{-1}\frac{1}{3}$ with the plane.

4. $2W\sin\alpha - Q$.

5. $\frac{1}{2}(P+Q)\operatorname{cosec}\alpha$; $\frac{P-Q}{P+Q}\tan\alpha$.

6. $\frac{(P-Q)^2}{\sqrt{4W^2-(P+Q)^2}}$; $\tan^{-1}\frac{P^2-Q^2}{4W^2-(P+Q)^2}$.

7. 14 pounds' weight.

8. $\frac{4}{5}\frac{2}{0}W$; $\frac{1}{7}\frac{1}{6}W$.

9. $\tan\alpha$, $2W\sin\alpha$; $W\sin 2\alpha$, at an angle α with the plane.

17. $W\tan(\alpha-\lambda)$.

18. $\alpha\pm\lambda$ from the highest and lowest points of the hoop.

20. $\frac{1}{2}W\sec(\alpha-\lambda)$.

32. It slips.

35. (i) $\frac{1}{3}W$, (ii) $\frac{5}{1}\frac{}{1}W$.

EXAMPLES XXV

2. $\frac{WP}{\sqrt{W^2+P^2}}$, $\tan^{-1}\frac{P}{W}$ with the horizontal.

3. $\tan^{-1}\frac{W}{P}$.

4. $\frac{1}{8}W\sqrt{3}$, $\frac{1}{8}W\sqrt{39}$ in a direction inclined at an angle $\tan^{-1}\frac{\sqrt{3}}{6}$ with the vertical.

5. $\frac{1}{4}W(\sqrt{3}-1)$.

6. ·164 . . .

7. $\tan^{-1}\frac{\sqrt{3}}{2}$, $\frac{1}{8}W\sqrt{19}$.

10. $w\left(\frac{l}{h}-2\right)$.

13. $\frac{1}{2}W\sqrt{5}$, $\frac{1}{2}W$, 45°.

15. $\cos^{-1}\left(\frac{1}{\mu}\sin\alpha-\cos\alpha\right)$.

18. 13·66 . . . pounds.

21. (i) $\tan^{-1}\frac{4}{3}$.
(ii) $\tan^{-1}\frac{13}{6}$.
(iii) $\frac{2}{3}W$.
(iv) $4W$.
(v) $\frac{9}{10}$ths of the way up the ladder.

22. $\tan^{-1}\cdot 3$.

23. To the top of the ladder.

24. It divides AB in the ratio $4:5$.

25. $\frac{3}{5}$ and $\frac{1}{2}$.

27. $115:21$.

28. Equilibrium is disturbed or not according as the load is placed above or below the centre of gravity of the ladder.

MISCELLANEOUS EXAMPLES

4. $a\cos\theta = b\sin^3\theta$; $W\operatorname{cosec}\theta$, $W\cot\theta$.

6. A force represented by $\frac{5}{2}AD$.

9. $4(AE)+2(BD)$.

24. A straight line.

36. The quadrilateral must be a rectangle.

44. The tensions of AD, BE, AF are respectively

$$\tfrac{1}{2}W\left(\frac{a\sec\alpha - l\tan\alpha}{a}\right),\ \tfrac{1}{2}W\sec\alpha,\ \tfrac{1}{2}W\tan\alpha\left(\frac{\sqrt{l^2-4al\sin\alpha+4a^2}}{a}\right).$$

48. The ratio must be less than 1.

52. $\dfrac{4l^2}{\sqrt{5l^2+2lb+b^2}}$.

55. In AB, BC, CD, DE, tensions $\frac{5}{8}W$, $\frac{5}{8}W$, $\frac{5}{8}W$, $\frac{5}{8}W$ respectively.
In BB', CC', DD', tensions $\frac{1}{4}W$, $\frac{1}{8}W$, $\frac{3}{8}W$ respectively.
In $B'C'$, $C'D'$, thrusts $\frac{5}{6}W$, $\frac{5}{8}W$ respectively.
In $B'C$, tension $\frac{1}{24}W\sqrt{34}$.
In AB', $C'D$, $D'E$, thrusts $\frac{1}{8}W\sqrt{34}$, $\frac{1}{24}W\sqrt{34}$, $\frac{1}{8}W\sqrt{34}$ respectively.

60. $\frac{1}{2}T\sqrt{7}$, $\frac{1}{2}T\sqrt{7}$, $\frac{3}{2}T$.

63. The vertical and horizontal components are $\frac{1}{2}W$ and $\frac{1}{3}W(6m-11)\sqrt{3}$ respectively.

64. At A, $\frac{1}{2}W\sqrt{5}$ in the line AO;
At C, $\frac{1}{2}W\sqrt{5}$ in the line CO;
where O divides BD in the ratio $3:1$.

66. $\dfrac{OD}{BD}\cdot T\cot\frac{1}{2}\alpha$ and $\dfrac{OB}{BD}\cdot T\cot\frac{1}{2}\alpha$;
where O is the point of intersection of the two strings.

71. $T=P+Q-S$ and $U=R+S-P$.

72. At the point of concurrence of the medians.

74. $\frac{3}{10}W\sqrt{5}$, $\frac{1}{10}W\sqrt{10}$ in the line BC.

75. $\dfrac{a\sqrt{27M^2+m^2}}{8M}$.

76. Each two must be together greater than the third.

79. $\frac{1}{2}\left\{\frac{a_1^2}{a_1+b}+\frac{a_2^2}{a_1+a_2+b}+\ldots+\frac{a_n^2}{\Sigma(a)+b}\right\}$

where $a_1>a_2>a_3>\ldots a_n$.

83. $\frac{1}{3}W$ each.

96. The apparent weight must be diminished by n times the weight of the portion broken off, where n is the ratio of the long arm to the short arm.

102. $2\tan^{-1}\mu$.

108. (i) The vertical angle must be less than $2\tan^{-1}2\mu$, where μ is the coefficient of friction.

(ii) The tension of the cord is between

$$\tfrac{1}{8}W\left(\frac{hl}{k^2}\right)-\tfrac{1}{2}\mu W\left(\frac{h}{k}\right) \text{ and } \tfrac{1}{8}W\left(\frac{hl}{k^2}\right)$$

according to the amount of friction called into play, where h, k, l are respectively the height of the joint above the ground, the height of the joint above the cord, and the length of the cord.

110. $(W+W')\{W(\cot\theta+\cot\theta')+(W+W')\cot\theta\}^{-\frac{1}{2}}$.

130. $\frac{\mu p(b+c)}{a+b+c}$.

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